University of Wisconsin
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Problem of the Week #3

(Spring 2017)

Let $f(x) = \sqrt{x^2+1}$. Prove: If $b \le -|a|$, then no tangent line to f passes through (a,b).

Solution:

Proof. Note first that for all x, we have

$$f(x) = \sqrt{x^2+1} > \sqrt{x^2} = |x|, \text{ and}$$

$$|f'(x)| = \left|\frac{x}{\sqrt{x^2+1}}\right| = \frac{|x|}{\sqrt{x^2+1}} < \frac{|x|}{\sqrt{x^2}} = 1.$$

Since f is differentiable on $(-\infty,\infty)$, f does not have a vertical tangent line.
Now let $b \le -|a|$ and $y_0 = \sqrt{x_0^2+1} > |x_0|$. Observe that $|a| + |x_0| \le -b + |x_0| < -b + y_0 = |b - y_0|$.
Also, by the triangle inequality, $|a - x_0| = |a + (-x_0)| \le |a| + |-x_0| = |a| + |x_0|$.
Therefore, the line $\mathscr{L}$ through (x_0, y_0) and (a,b), if it is not vertical, has slope

$$\left|\frac{b-y_0}{a-x_0}\right| > \left|\frac{|a|+|x_0|}{a-x_0}\right| = \frac{|a|+|x_0|}{|a-x_0|} \ge \frac{|a|+|x_0|}{|a|+|x_0|} = 1.$$

Thus $\mathscr{L}$ is not tangent to f at (x_0, y_0). □

Source: Mark, Melissa, and Michael Schramm. "When You Wander off on a Tangent, Where Do You End Up?" *College Mathematics Journal* **47**:5 (November 2016), 334-339.