University of Wisconsin Platteville
Department of Mathematics

Problem of the Week #2

(Spring 2017)

Let N be a positive integer. Our job is to pick integers $1 = a_1 < a_2 < \cdots < a_{10} = N$, satisfying $a_k \geq a_{k-1} + k$ for $2 \leq k \leq 10$. In how many different ways can we do this job?

Solution:
There are $\binom{N-47}{8}$ ways to choose $(a_1, \ldots, a_{10})$.

Proof. Let $1 = b_1 < b_2 < \cdots < b_{10} = N - 45$. For each k, define $a_k = b_k + (1 + \cdots + k - 1))$.
Then $a_1 = b_1 = 1$, and $a_{10} = b_{10} + (1 + 2 + \cdots + 9) = b_{10} + 45 = N$. Moreover, given $2 \leq k \leq 10$:

$$\begin{aligned} b_k &> b_{k-1} \\ b_k &\geq b_{k-1} + 1 \\ a_k - (1 + 2 + \cdots + (k-1)) &\geq a_{k-1} - (1 + 2 + \cdots + (k-2)) + 1 \\ a_k - (k-1) &\geq a_{k-1} + 1 \\ a_k &\geq a_{k-1} + k \end{aligned}$$

Since every step above is reversible, we have exactly one way to do our job for every way of choosing $2 \leq b_2 < b_3 < \cdots < b_9 \leq N - 46$, and there are $\binom{N-47}{8}$ ways to do this. □

Source: Dan Swenson, Black Hills State University.