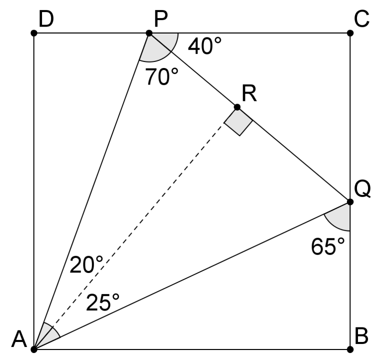
University of Wisconsin
PLATTEVILLE
Department of Mathematics

Problem of the Week #1

(Spring 2017)

Triangle $\triangle APQ$ is inscribed in the square $ABCD$ as shown in the figure, so that $\angle PAQ = 45°$ and $\angle AQB = 65°$. Find the measure of angle $\angle CPQ$.



Solution:
$\angle CPQ = 40°$.

Proof. We know $\angle BAQ = 25°$, so $\angle DAP = 20°$. Let R_1 be the reflection of B across AQ, and let R_2 be the reflection of D across AP.
I claim that $R_1 = R_2$. Note that $\angle QAR_1 + \angle PAR_2 = 25° + 20° = 45° = \angle PAQ$, so R_1, R_2, and A are collinear. But $|AR_1| = |AB| = |AD| = |AR_2|$, which proves the claim. Let $R = R_1 = R_2$. Now $\angle PRA + \angle ARQ = 90° + 90° = 180°$, so R lies on PQ. It follows that $\angle APR = \angle APD = 70°$, so $\angle CPQ = (180 - 70 - 70)° = 40°$. □

Alternate proof. Suppose $|AB| = 1$. Then $P = (\tan 20°, 1)$ and $Q = (1, \tan 25°)$. Let $\vec{v} = \overrightarrow{PQ} = \langle 1 - \tan 20°, \tan 25° - 1\rangle$. Now let $\theta = \angle CPQ$. We have $\vec{v} \cdot \vec{\imath} = |\vec{\imath}||\vec{v}|\cos\theta$, so $\theta = \arccos \frac{1-\tan 20°}{\sqrt{(1-\tan 20°)^2+(\tan 25°-1)^2}} \approx 40°$. □

Source: Park, Poo-Sung. “Quickies 1063.” *Mathematics Magazine* **89**:4 (October 2016), pp. 294, 300.