UNIVERSITY OF WISCONSIN
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #10

(Fall 2023)

I took a big bag of tokens to a party, containing some tokens worth 1 point, some worth 2 points, and some worth -1 point. Ben drew some tokens, and so did Cinda.
Amazingly, it turned out that Ben and Cinda each drew an assortment of tokens for which:

1. the values added up to 19; and
2. the squares of the values added up to 99.

But given those remarkable coincidences, it was even more stunning that the cubes of Ben's token values added up to the smallest possible total m, while the cubes of Cinda's token values added up to the greatest possible total M. Find $\dfrac{M}{m}$.

Solution:

Suppose that either Ben or Cinda drew a tokens worth -1, b tokens worth 1, and c tokens worth 2. We are given: $\begin{cases} -a+b+2c &= 19, \\ a+b+4c &= 99. \end{cases}$ Therefore:

$$\begin{aligned} (a+b+4c)-2(-a+b+2c) &= 99-2(19) \\ 3a-b &= 61 \\ &\boxed{b=3a-61} \\ -a+(3a-61)+2c &= 19 \\ 2a+2c &= 80 \\ &\boxed{c=40-a}. \end{aligned}$$

We know $b \geq 0 \Rightarrow a \geq \frac{61}{3}$, while $c \geq 0 \Rightarrow 40 \geq a$. Since a is an integer, $21 \leq a \leq 40$.
Finally, the sum of the cubes of the tokens' values is:

$$-a+b+8c = -a+(3a-61)+8(40-a) = 259-6a.$$

So Ben's minimum value was $m = 259-6(40) = 19$, with $(a,b,c) = (40,59,0)$, and Cinda's maximum value was $M = 259-6(21) = 133$, with $(a,b,c) = (21,2,19)$. Therefore $\frac{M}{m} = \frac{133}{19} = \boxed{7}$.

Source: Problem 28, 1999 American High School Mathematics Examination, with solution, from "Art of Problem Solving" [https://artofproblemsolving.com/wiki/index.php/1999_AHSME_Problems/Problem_28].