University of Wisconsin
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #9

(Fall 2023)

Which positive integers N have the property that when you delete the third digit of N (in base ten, counting from the left), you get a divisor of N?

Solution:

Suppose that M is formed by deleting the third digit of N. If N consists of two digits followed by one or more zeros, then $N = 10M$, because then deleting the third digit of N, a zero, is the same as dividing N by 10, and so M is a divisor of N.
Now suppose that $N = kM$ for some integer k. In base ten, we can write

$$N = d_0 \cdot 10^n + d_1 \cdot 10^{n-1} + d_2 \cdot 10^{n-2} + d_3 \cdot 10^{n-3} + \cdots + d_{n-1} \cdot 10 + d_n \quad (1)$$
$$M = d_0 \cdot 10^{n-1} + d_1 \cdot 10^{n-2} + d_3 \cdot 10^{n-3} + \cdots + d_{n-1} \cdot 10 + d_n$$
$$10M = d_0 \cdot 10^n + d_1 \cdot 10^{n-1} + d_3 \cdot 10^{n-2} + \cdots + d_{n-2} \cdot 10^2 + d_n \cdot 10 \quad (2)$$

We can subtract (1) from (2) to obtain:

$$10M - N = (d_3 - d_2) \cdot 10^{n-2} + (d_4 - d_3) \cdot 10^{n-3} + \cdots + (d_n - d_{n-1}) \cdot 10 - d_n$$
$$(10 - k)M = (d_3 - d_2) \cdot 10^{n-2} + (d_4 - d_3) \cdot 10^{n-3} + \cdots + (d_n - d_{n-1}) \cdot 10 - d_n \quad (3)$$

Hence $\frac{10-k}{k}N < 10^{n-1}$. If we suppose that $k < 10$, though, we find

$$\frac{10-k}{k}N \geq \frac{1}{k}N > \frac{1}{10}N = d_0 \cdot 10^{n-1} + d_1 \cdot 10^{n-2} + d_2 \cdot 10^{n-3} + d_3 \cdot 10^{n-4} + \cdots + d_{n-1} + \frac{d_n}{10} > 10^{n-1},$$

which contradicts (3).
Likewise, if $k \geq 11$, then $\frac{k-10}{k}N < 10^{n-1}$. If $k \geq 12$, this implies $10k = k + 9k \geq k + 108 > k + 100$, so $10(k-10) > k$, which means $\frac{k-10}{k} > \frac{1}{10}$. Now:

$$\frac{k-10}{k}N > \frac{1}{10}N = d_0 \cdot 10^{n-1} + d_1 \cdot 10^{n-2} + d_2 \cdot 10^{n-3} + d_3 \cdot 10^{n-4} + \cdots + d_{n-1} + \frac{d_n}{10} > 10^{n-1},$$

which is a contradiction again.
We can't even have $k = 11$, or else $M = \frac{N}{11} = \frac{k-10}{k}N < 10^{n-1}$, which would mean that M had two fewer digits than N, though we only deleted one digit from N to form M.
Therefore $k = 10$, and from (3), we know

$$0 = (d_3 - d_2) \cdot 10^{n-2} + (d_4 - d_3) \cdot 10^{n-3} + \cdots + (d_n - d_{n-1}) \cdot 10 + (0 - d_n),$$

implying that $d_2 = d_3 = d_4 = \cdots = d_n = 0$. □

Source: D.O. Shklarsky, N.N. Chentzov, and I.M. Yaglom, *The USSR Olympiad Problem Book: Selected Problems and Theorems of Elementary Mathematics*, Dover Publications (1993), 11, 105–106.