UNIVERSITY OF WISCONSIN
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #7

(Fall 2023)

Find real numbers a and b for which:

$$\begin{cases} a^4 + 8b = 4(a^3 - 1) - 16\sqrt{3}, \\ b^4 + 8a = 4(b^3 - 1) + 16\sqrt{3}. \end{cases}$$

Solution:

First notice that $a \neq b$, because $a^4 + 8b - 4(a^3 - 1) \neq b^4 + 8a - 4(b^3 - 1)$.
Now add the given equations:

$$\begin{aligned} a^4 + 8b + b^4 + 8a &= 4a^3 - 4 + 4b^3 - 4 \\ (a^4 - 4a^3 + 8a + 4) + (b^4 - 4b^3 + 8b + 4) &= 0 \\ (a^2 - 2a - 2)^2 + (b^2 - 2b - 2)^2 &= 0 \end{aligned}$$

Perfect squares can't be negative, so a and b must be the two solutions of the quadratic equation $x^2 - 2x - 2 = 0$. By the quadratic formula, $\{a, b\} = \left\{1 + \sqrt{3}, 1 - \sqrt{3}\right\}$. Finally, we check that $\boxed{a = 1 + \sqrt{3}}$ and $\boxed{b = 1 - \sqrt{3}}$, because

$$\begin{aligned} &\left(1 + \sqrt{3}\right)^4 + 8\left(1 - \sqrt{3}\right) - 4\left(\left(1 + \sqrt{3}\right)^3 - 1\right) \\ &= 1 + 4\sqrt{3} + 18 + 12\sqrt{3} + 9 + 8 - 8\sqrt{3} - 4 - 12\sqrt{3} - 36 - 12\sqrt{3} + 4 \\ &= (1 + 18 + 9 + 8 - 4 - 36 + 4) + (4 + 12 - 8 - 12 - 12)\sqrt{3} \\ &= -16\sqrt{3}. \end{aligned}$$

Source: Titu Andreescu and Jonathan Kane, *Purple Comet! Math Meet: The First Ten Years*, XYZ Press (2013), 109.