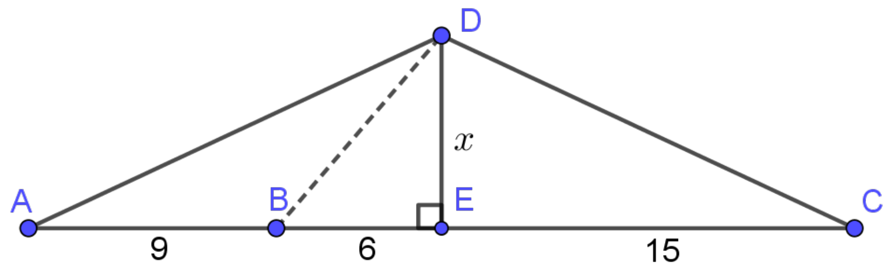
University of Wisconsin
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #5

(Fall 2023)

Point B is on segment $\overline{AC}$ with $AB = 9$ and $BC = 21$. Point D is not on $\overline{AC}$, and $AD = CD$. Assuming that AD and BD are integers, what could be the perimeter of $\triangle ACD$?



Solution:
The possible perimeters of $\triangle ACD$ are 64, 96, and 220.

Solution. Let P denote the perimeter of $\triangle ACD$: we have $P = 30 + AD + CD = 30 + 2 \cdot AD$. Construct the altitude $\overline{DE} \perp \overline{AC}$, with length $x > 0$. Since $\triangle ADE$ is isosceles, $AE = 15 = CE$, and $BE = 15 - 9 = 6$. The Pythagorean theorem, applied to $\triangle ADE$ and $\triangle BDE$, yields:

$$\begin{aligned} 225 &= AD^2 - x^2 \\ \underline{36} &= \underline{BD^2 - x^2} \\ 189 &= AD^2 - BD^2 \\ 189 &= (AD - BD)(AD + BD) \end{aligned}$$

Since $0 < AD - BD < AD + BD$, we know $AD - BD$ is a positive divisor of $3^3 \cdot 7 = 189$ with $1 \leq AD - BD < \sqrt{189} < 14$. Therefore $AD - BD$ is in $\{1, 3, 7, 9\}$. Observe that the perimeter of $\triangle ACD$ is $P = 30 + 2 \cdot AD = 30 + (AD + BD) + (AD - BD)$, and consider all four cases.

- If $AD - BD = 1$, then $AD + BD = 189$ and $P = 30 + 189 + 1 = 220$.
- If $AD - BD = 3$, then $AD + BD = 63$ and $P = 30 + 63 + 3 = 96$.
- If $AD - BD = 7$, then $AD + BD = 27$ and $P = 30 + 27 + 7 = 64$.
- If $AD - BD = 9$, then $AD + BD = 21$. But that means $AD = 15$ and $BD = 6$; in other words, $D = E$, and D lies on AC, contrary to our initial assumption.

Thus P is either 64, 96, or 220. □

Source: AIME 2003, Problem #7. In: Scott A. Annin. *A Gentle Introduction to the American Invitational Mathematics Exam.* MAA Press (2015), 160–161.