University of Wisconsin
PLATTEVILLE
Department of Mathematics

Problem of the Week #4

(Fall 2023)

In the binomial expansion of $(2+3)^{30}$, which term is the largest?

Solution:
The term $\binom{30}{18}2^{12}3^{18} = 137{,}253{,}877{,}861{,}118{,}054{,}400$ is largest.

Solution. By the binomial theorem, $(2+3)^{30} = \sum_{k=0}^{30}\binom{30}{k}2^{30-k}3^k$.

Suppose the largest term in this sum occurs when $k=r$. Then the r^{th} term is greater than or equal to the term on either side of it:

$$\begin{aligned}
\binom{30}{r}2^{30-r}3^r &\geq \binom{30}{r-1}2^{31-r}3^{r-1}, \\
\frac{30!}{r!(30-r)!}2^{30-r}3^r &\geq \frac{30!}{(r-1)!(31-r)!}2^{31-r}3^{r-1}, \\
\frac{3}{r} &\geq \frac{2}{31-r}, \\
93-3r &\geq 2r, \\
93 &\geq 5r,
\end{aligned}
\qquad
\begin{aligned}
\binom{30}{r}2^{30-r}3^r &\geq \binom{30}{r+1}2^{29-r}3^{r+1} \\
\frac{30!}{r!(30-r)!}2^{30-r}3^r &\geq \frac{30!}{(r+1)!(29-r)!}2^{29-r}3^{r+1} \\
\frac{2}{30-r} &\geq \frac{3}{r+1} \\
2r+2 &\geq 90-3r \\
5r &\geq 88.
\end{aligned}$$

Since $5r$ is a multiple of 5 between 88 and 93, $5r = 90$ and $r = 18$. □

Source: Charles F. Pinzka, "Problem solving and some problems." In: National Council of Teachers of Mathematics, *Enrichment Mathematics for High School: Twenty-Eighth Yearbook* (1963), pp. 178–183.