University of Wisconsin
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #3

(Fall 2023)

You're about to play a game. On each turn, you'll draw a number at random from a hat, then make a choice. If you draw the number k, then you may collect k dollars, or you may instead add to the hat the k smallest positive integers that aren't already in the hat. For example, if your turn begins with the numbers 1 through 8 in the hat, and you draw the number 3, you may either collect \$3 or add the numbers 9, 10, and 11 to the hat.

When the game starts, only the number 1 is in the hat. If you get to play for 100 turns, and you choose your strategy to maximize the amount of money you collect, on average how much money can you expect to win?

Solution:

You can expect to win $2\left(\frac{3}{2}\right)^{98}$ (or about 361.4 quadrillion) dollars.

Proof. Let $f(t,n)$ denote the expected value of the remainder of the game if you have t turns left and the numbers 1 through n are in the hat. It is clear that $f(0,n)=0$. We will prove by induction on t that $f(t,n)=c_t(n+1)$ for some c_t that does not depend on n. This is obvious for $t=0$, taking $c_0=0$.

Let $t\geq 0$ such that $f(t,n)=c_t(n+1)$. Suppose now that with $t+1$ turns left to go, you draw the number k from the hat. If you choose to collect \$$k$, your expected winnings in the last $t+1$ turns are $k+f(t,n)=k+c_t(n+1)$. On the other hand, if you add numbers to the hat, you can expect to win $f(t,n+k)=c_t(n+k+1)=c_tk+c_t(n+1)$.

So if $c_t\leq 1$, then for any k, (because $k>0$) your best play is to take the money, and

$$f(t+1,n)=\frac{1}{n}\sum_{k=1}^{n}\left[k+c_t(n+1)\right]=\frac{1}{n}\left[\frac{n(n+1)}{2}+n\cdot c_t(n+1)\right]=\left[\frac{1}{2}+c_t\right](n+1).$$

But if $c_t\geq 1$, then your best play is to add numbers to the hat, and

$$f(t+1,n)=\frac{1}{n}\sum_{k=1}^{n}\left[c_tk+c_t(n+1)\right]=\frac{1}{n}\left[\frac{n(n+1)}{2}+n\cdot(n+1)\right]c_t=\frac{3}{2}c_t(n+1).$$

This completes the induction, and shows that $c_{t+1}=\begin{cases}\frac{1}{2}+c_t, & c_t\leq 1,\\ \frac{3}{2}c_t, & c_t\geq 1.\end{cases}$

We can now write down the sequence $\{c_t\}=\left\{0,\frac{1}{2},1,\frac{3}{2},\frac{9}{4},\frac{27}{8},\ldots\right\}$, or $c_t=\begin{cases}t/2, & t\leq 3,\\ (3/2)^{t-2}, & t\geq 2.\end{cases}$

The expected value of the entire game is $f(100,1)=c_{100}(1+1)=2\cdot\left(\frac{3}{2}\right)^{98}$. □

Remark. Depending on how you count, this is roughly 4000 times as much money as there is in the world.

Source: Yonah Borns-Weil. In: Zach Wissner-Gross. "How Much Money Can You Pull Out of a Hat?" `fivethirtyeight.com/features/how-much-money-can-you-pull-out-of-a-hat/`