UNIVERSITY OF WISCONSIN
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #2

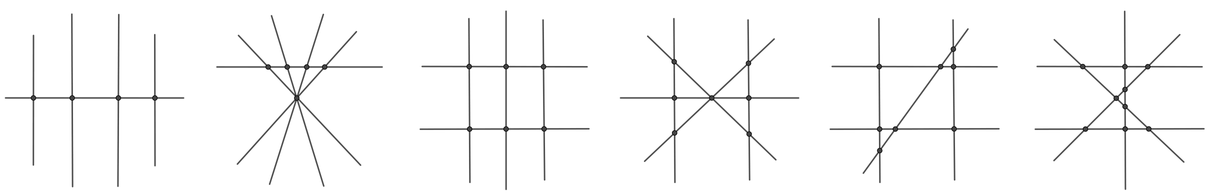
(Fall 2023)

Five distinct lines in the plane can intersect (two at a time) in as many as ten different points. On the other hand, five parallel lines would have zero intersection points.
Find the largest integer $k < 10$ for which no set of five distinct lines in the plane intersects (two at a time) in exactly k points.

Solution:

The desired integer is $k = 3$.

Proof. If $3 < n < 10$, then five distinct lines in the plane can intersect (two at a time) in exactly n points, as shown here:



Now suppose for the sake of contradiction that we have five distinct lines in the plane with exactly 3 intersection points A, B, and C. Some of our lines might be parallel, but we can't have pairs of parallel lines in two different directions; if we did, they'd form a parallelogram, and there would be at least 4 intersection points. So any pairs of parallel lines that we do have must all run in the same direction. For convenience, let's declare that if two or more of our lines are parallel, then all of those lines pass through a fictional point called ∞. With this agreement, every pair of distinct lines has a unique intersection point, which is either A, B, C, or ∞.
If all five of our lines passed through the same intersection point, that would be the only intersection point (since if two of our lines also intersected at another intersection point, they would be the same line). So that can't happen; instead, since each of our lines intersects all four of the others, each of our lines must pass through at least two of $\{A, B, C, \infty\}$. Thus our five lines must all be members of the set $\{\overline{AB}, \overline{AC}, \overline{BC}, \overline{A\infty}, \overline{B\infty}, \overline{C\infty}\}$. (So we must have at least one pair of parallel lines: without ∞, we'd only have 3 possible lines.)
If three of our four intersection points were on the same line, then three of our six possible lines would be identical, leaving only four distinct lines in our set of possibilities. That contradicts the assumption that we started with five distinct lines, so no three intersection points are collinear.
On the other hand, suppose without loss of generality that $\overline{AB}$ and $\overline{C\infty}$ are two of our five lines. If they intersect at A or B, then we have three collinear intersection points lying on $\overline{C\infty}$. Otherwise they intersect at C or ∞, and then we have three collinear intersection points lying on $\overline{AB}$.
In either case, we have a contradiction. We conclude that five distinct lines in the plane cannot have exactly 3 intersection points. $\square$

Source: ShapiroA, "LL One-Day Special: Pen and Paper Math 8," `LearnedLeague.com`.