UNIVERSITY OF WISCONSIN
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

# Problem of the Week #10

(Fall 2022)

Define $F(x,y) = x + y + x^2y + xy^2 + x^3y^2 + x^2y^3 + x^4y^3 + x^3y^4 + \dots$.

Show that if $x$, $y$, and $z$ are real numbers with absolute values less than $\sqrt{2} - 1$, then $F(x, F(y,z)) = F(F(x,y), z)$.

**Solution:**

*Proof.* Factor by grouping to obtain $F(x,y) = (x+y)\,(1 + xy + x^2y^2 + x^3y^3 + \dots)$.
When $|xy| < 1$, this geometric series converges, and $F(x,y) = \dfrac{x+y}{1-xy}$.
Thus, for any angles $\theta$ and $\varphi$ with $|\tan\theta\tan\varphi| < 1$, we have

$$F(\tan\theta, \tan\varphi) = \frac{\tan\theta + \tan\varphi}{1 - \tan\theta\tan\varphi} = \tan(\theta + \varphi).$$

Also, by a half-angle formula, we know that $\tan\frac{\pi}{8} = \frac{1-\cos\frac{\pi}{4}}{\sin\frac{\pi}{4}} = \frac{1-(1/\sqrt{2})}{1/\sqrt{2}} = \sqrt{2} - 1$.
Therefore, given $x, y, z$ with absolute values less than $\sqrt{2} - 1$, we can let $\alpha = \arctan x$, $\beta = \arctan y$, and $\gamma = \arctan z$, and we know that $\alpha, \beta, \gamma \in \left(-\frac{\pi}{8}, \frac{\pi}{8}\right)$. It follows that $\alpha$, $\gamma$, $\alpha + \beta$, and $\beta + \gamma$ are all in the interval $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$, which means their tangents are in $(-1, 1)$, and so are all products of those tangents. Finally:

$$\begin{aligned}
F(x, F(y,z)) &= F(\tan\alpha, F(\tan\beta, \tan\gamma)) \\
&= F(\tan\alpha, \tan(\beta + \gamma)) \\
&= \tan[\alpha + (\beta + \gamma)] \\
&= \tan[(\alpha + \beta) + \gamma] \\
&= F(\tan(\alpha + \beta), \tan\gamma) \\
&= F(F(\tan\alpha, \tan\beta), \tan\gamma) \\
&= F(F(x,y), z).
\end{aligned}$$

□