Problem of the Week #9

(Fall 2022)

Given real numbers $a_0 > b_0 > 0$, construct a pair of sequences inductively as follows.
For each n, let a_{n+1} be the arithmetic mean of a_n and b_n:

$$a_{n+1} = \frac{a_n + b_n}{2}.$$

And let b_{n+1} be the harmonic mean of a_n and b_n:

$$\frac{1}{b_{n+1}} = \frac{\frac{1}{a_n} + \frac{1}{b_n}}{2}.$$

Show that the sequences $\{a_n\}$ and $\{b_n\}$ both converge to the same limit, which is $\sqrt{a_0 b_0}$, the geometric mean of a_0 and b_0.

Solution:

Proof. Define $A(x,y) = \dfrac{x+y}{2}$, $H(x,y) = \dfrac{1}{\left(\frac{\frac{1}{x}+\frac{1}{y}}{2}\right)} = \dfrac{2xy}{x+y}$, and $G(x,y) = \sqrt{xy}$ — the arithmetic, harmonic, and geometric means of x and y, respectively.
We are given $a_{n+1} = A(a_n, b_n)$ and $b_{n+1} = H(a_n, b_n)$. Note also that

$$G(a_{n+1}, b_{n+1}) = \sqrt{\frac{a_n + b_n}{2} \cdot \frac{2a_n b_n}{a_n + b_n}} = \sqrt{a_n b_n} = G(a_n, b_n),$$

which therefore equals $G(a_0, b_0) = \sqrt{a_0 b_0}$ by induction.
It is well known* that when $y > x > 0$, we have $x < H(x,y) < G(x,y) < A(x,y) < y$, so

$$b_n < b_{n+1} < \sqrt{a_0 b_0} < a_{n+1} < a_n.$$

This shows that $\{a_n\}$ is a decreasing sequence with lower bound $\sqrt{a_0 b_0}$, and $\{b_n\}$ is an increasing sequence with upper bound $\sqrt{a_0 b_0}$, so both sequences converge by the monotone convergence theorem. Let $\alpha = \lim_{n\to\infty} a_n$ and $\beta = \lim_{n\to\infty} b_n$. Since A is a continuous function,

$$\frac{\alpha+\beta}{2} = A(\alpha, \beta) = A\left(\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n\right) = \lim_{n\to\infty} A(a_n, b_n) = \lim_{n\to\infty} a_{n+1} = \alpha,$$

so $\alpha + \beta = 2\alpha$, or $\beta = \alpha$. Since $\{a_n\}$ and $\{b_n\}$ have the same limit, it must be $\sqrt{a_0 b_0}$. $\square$

Source: Suggested by Dan Swenson; see D.M.E. Foster and G.M. Phillips, "A generalization of the Archimedean double sequence," *Journal of Mathematical Analysis and Applications* **101**:2 (July 1984), 575–581.

*Among people who know about it.