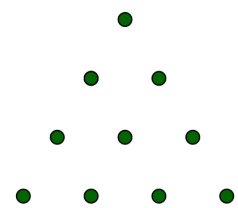
UNIVERSITY OF WISCONSIN
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #7
(Fall 2022)

If N dots can be arranged in rows to form an equilateral triangle, with one dot in the first row, two dots in the second row, and in general k dots in the k^{th} row, then N is called a *triangular number*. For instance, ten is a triangular number (ten dots form a 4-row triangle).



In base 9, ten is written 11. Show that every number whose base-9 digits are all 1s is triangular.

Solution:

Proof. If T_n denotes the triangular number represented by a triangle with n rows, then

$$T_n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Now let a_j be the number whose base-9 digits are j consecutive 1s, so in base 9

$$\{a_j\} = \{1, 11, 111, 1111, \ldots\}.$$

Given $a_k = 11\cdots1$ in base 9, we have $9a_k = 11\cdots10$, and $9a_k + 1 = 11\cdots11 = a_{k+1}$.

It is clear that $a_1 = 1 = T_1$, so a_1 is triangular. Now suppose for induction that a_k is triangular. There is some n with $a_k = T_n = \frac{n(n+1)}{2}$, and

$$a_{k+1} = 9a_k + 1 = 9\frac{n(n+1)}{2} + 1 = \frac{9n(n+1)+2}{2} = \frac{9n^2+9n+2}{2} = \frac{(3n+1)(3n+2)}{2} = T_{3n+1},$$

so a_{k+1} is triangular. By induction, a_j is triangular for all $j \geq 1$. □

Source: G. W. Wishard. *American Mathematical Monthly* **39**:3 (March 1932), 179.