UNIVERSITY OF WISCONSIN
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #5

(Fall 2022)

Let z be a complex number. Given that $z + \frac{1}{z} = 2\cos(5^\circ)$, find $z^{2022} + \frac{1}{z^{2022}}$.

Solution:

Choose real numbers r and θ so that $r > 0$ and $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$, which means $\frac{1}{z} = \frac{1}{r}e^{-i\theta} = \frac{1}{r}(\cos\theta - i\sin\theta)$. We are given:

$$2\cos\left(\frac{\pi}{36}\right) = z + \frac{1}{z} = r\left[\cos\theta + i\sin\theta\right] + \frac{1}{r}\left[\cos\theta - i\sin\theta\right] = \cos\theta\left[r + \frac{1}{r}\right] + i\sin\theta\left[r - \frac{1}{r}\right].$$

The imaginary parts of this equation are equal, so $0 = (\sin\theta)\left[r - \frac{1}{r}\right]$. Suppose that $\sin\theta = 0$.

- If $\cos\theta = 1$, then $2\cos\frac{\pi}{36} = r + \frac{1}{r}$, so $r^2 - \left(2\cos\frac{\pi}{36}\right)r + 1 = 0$. The discriminant of this quadratic equation is $4\cos^2\frac{\pi}{36} - 4 < 0$, so r is not real — a contradiction.
- If $\cos\theta = -1$, then $2\cos\frac{\pi}{36} = -\left[r + \frac{1}{r}\right]$. The left side of this equation is positive and the right side is negative — a contradiction.

Since $\sin\theta \neq 0$, $r = \frac{1}{r}$; but $r > 0$, so $r = 1$. Now $2\cos\left(\frac{\pi}{36}\right) = 2\cos\theta + 0i\sin\theta$, so $\cos\left(\frac{\pi}{36}\right) = \cos\theta$. We may require $-\pi < \theta \leq \pi$, and conclude that $\theta = \pm\frac{\pi}{36}$ and $z = e^{\pm\pi i/36}$. Finally,

$$\begin{aligned}
z^{2022} + \frac{1}{z^{2022}} &= e^{\pm 2022\pi i/36} + e^{\mp 2022\pi i/36} \\
&= e^{\pm 337\pi i/6} + e^{\mp 337\pi i/6} \\
&= \cos\left(\frac{337\pi}{6}\right) + i\sin\left(\frac{337\pi}{6}\right) + \cos\left(-\frac{337\pi}{6}\right) + i\sin\left(-\frac{337\pi}{6}\right) \\
&= 2\cos\left(\frac{337\pi}{6}\right) \\
&= 2\cos\left(56\pi + \frac{\pi}{6}\right) \\
&= 2\left(\frac{\sqrt{3}}{2}\right) = \boxed{\sqrt{3}}.
\end{aligned}$$

Source: Adapted from Problem #9 of the 2000 American Invitational Mathematics Exam (Version 2). In: Scott A. Annin. *A Gentle Introduction to the American Invitational Mathematics Exam.* The Mathematical Association of America (2015), 136-137.