Problem of the Week #4

(Fall 2022)

In my head-to-head trivia league, each player competes exactly once per season against every other player, earning two points per win plus one point per draw.
Last season we found that in any group of three players, there was someone who scored exactly three points in their matches against the other two.
What is the largest number of players that could have been in my league last season?

Solution:

The league contained at most five players.

Proof. Draw a graph with a vertex for each player and an edge joining each pair of players. Paint each edge red if the match between the corresponding players was a draw, and paint it blue otherwise.
Suppose that this graph has at least six vertices. It is known that such a graph must contain either a red triangle or a blue triangle. If there is a red triangle, then each of the corresponding players scored exactly two points against the other two. If there is a blue triangle, then each of the corresponding players scored an even number of points against the other two, since there were no draws among this group. In either case, none of the three corresponding players scored exactly three points against the other two, contrary to our initial assumption. Thus the graph can't have more than five vertices, and the league can't have more than five players.
But it is possible that the league contained exactly five players. For example, it is possible that A beat B, who beat C, who beat D, who beat E, who beat A, and the other five matches were ties. Then in any group of three players, there are two who did not tie. Label the three players x, y, and z in such a way that x beat y. Then x did not also beat z. If x and z tied, then x has exactly three points against y and z. Otherwise, z beat x, and y and z tied, so z has exactly three points against x and y. □

Remark. How do we know that the graph contains either a red triangle or a blue triangle? With six or more vertices, there are at least five edges coming out of any vertex v. If at least 3 of those edges are red, then any red edge between two of the other endpoints would complete a red triangle, but if there isn't such a red edge, then the three other endpoints are the vertices of a blue triangle. On the other hand, if there aren't at least 3 red edges coming out of v, then there are at least 3 blue edges there instead, and the same reasoning applies.

Source: "Chess tournament with 1.5 points winners," `https://www.cut-the-knot.org/m/Algebra/CC124Crux.shtml`.