UNIVERSITY OF WISCONSIN
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #3

(Fall 2022)

Which is larger: 10000^{10000}, or 10001^{9999}?

Solution:

10000^{10000} is larger.

Proof. We know that $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$.

Let $N = 10000$. Since N is large,

$$\frac{(N+1)^{N-1}}{N^N} = \frac{1}{N+1} \cdot \frac{(N+1)^N}{N^N} = \frac{1}{N+1} \cdot \left(\frac{N+1}{N}\right)^N = \frac{1}{N+1} \cdot \left(1 + \frac{1}{N}\right)^N \approx \frac{1}{N+1} \cdot e = \frac{e}{10001} < 1.$$

Therefore $(N+1)^{N-1} < N^N$; that is, $10001^{9999} < 10000^{10000}$. □

Remark. This was a little sketchy, because of the "≈" along the way. To clean up, let $f(x) = \left(1 + \frac{1}{x}\right)^x$. Using logarithmic differentiation, we can show that

$$\begin{aligned} f'(x) &= f(x) \left[\ln\left(1 + \frac{1}{x}\right) - \frac{1}{x+1}\right] \\ &= f(x) \left[\ln(x+1) - \ln(x) - \frac{1}{x+1}\right] \\ &= f(x) \left[\int_x^{x+1} \frac{1}{t}\, dt - \frac{1}{x+1}\right] \\ &= f(x) \int_x^{x+1} \left(\frac{1}{t} - \frac{1}{x+1}\right) dt, \end{aligned}$$

which is positive, because $f(x) > 0$ when x is real and $\frac{1}{t} - \frac{1}{x+1} > 0$ when $t < x + 1$. Since $f'(x) > 0$, we know $f(x)$ is increasing, so

$$\left(1 + \frac{1}{N}\right)^N = f(N) < \lim_{x \to \infty} f(x) = e,$$

which is what we needed.