UNIVERSITY OF WISCONSIN
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #9

(Fall 2021)

Let $\{a_n\}$ be a sequence of positive integers with

$$\frac{a_1}{a_2} = \frac{a_1 + a_2}{a_3 + a_4} = \frac{a_1 + a_2 + a_3}{a_4 + a_5 + a_6} = \dots$$

Prove that a_2 is divisible by a_1.

Solution:

Proof. Let $s_n = a_1 + a_2 + \cdots + a_n$, and let $C = \frac{a_1}{a_2}$. We know that for $k \geq 1$:

$$\begin{aligned} \frac{s_k}{s_{2k} - s_k} &= C \\ s_k &= Cs_{2k} - Cs_k \\ (C+1)s_k &= Cs_{2k} \\ (1 + \tfrac{1}{C})s_k &= s_{2k} \end{aligned}$$

Let $q = 1 + \frac{1}{C} = \frac{s_2}{s_1}$, and subtract the equations $\begin{cases} qs_n &= s_{2n} \\ qs_{n-1} &= s_{2n-2} \end{cases}$ to obtain $qa_n = a_{2n-1} + a_{2n}$.
Specifically, $qa_1 = a_1 + a_2$, so $a_2 = (q-1)a_1$, which means we want to show that q is an integer. Observe that q is rational, and write $q = \frac{h}{k}$ in lowest terms. This yields $ha_n = k(a_{2n-1} + a_{2n})$, and since h is relatively prime to k, a_n is a multiple of k.
Now let $b_n = \frac{a_n}{k}$. Then $\{b_n\}$ is a new sequence of positive integers with the property that

$$C = \frac{b_1}{b_2} = \frac{b_1 + b_2}{b_3 + b_4} = \frac{b_1 + b_2 + b_3}{b_4 + b_5 + b_6} = \dots$$

and consequently the work we've just done shows that every b_n is divisible by k, which means a_n is divisible by k^2. By repeating this process m times, we conclude that a_n is divisible by k^m for every $m \geq 1$. Thus $k = 1$, so q is an integer. □

Remark. The problem is inspired by a fact about the odd integers that Galileo pointed out:

$$\frac{1}{3} = \frac{1+3}{5+7} = \frac{1+3+5}{7+9+11} = \dots$$

Source: Kenneth O. May. "Galileo sequences, a good dangling problem." *The American Mathematical Monthly* **79**:1 (January 1972), 67–69.