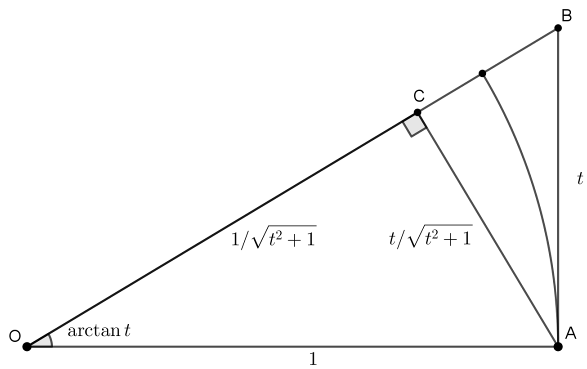
UNIVERSITY OF WISCONSIN
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #8

(Fall 2021)

Prove that if $t > 0$, then $(\arctan t)\left(\arctan \frac{1}{t}\right) > \frac{t}{2(t^2+1)}$.

Solution:



It is enough to prove the inequality for $0 < t \le 1$. If we accomplish this, then for $t > 1$, we can let $u = 1/t$, so $0 \le u < 1$ and

$$(\arctan u)\left(\arctan \frac{1}{u}\right) > \frac{u}{2(u^2+1)}$$
$$\left(\arctan \frac{1}{t}\right)(\arctan t) > \frac{1/t}{2((1/t^2)+1)} \cdot \frac{t^2}{t^2}$$
$$(\arctan t)\left(\arctan \frac{1}{t}\right) > \frac{t}{2(1+t^2)}.$$

So suppose that $0 < t \le 1$, so $\frac{1}{t} > 1$. Note that $\arctan t > 0$ and $\arctan \frac{1}{t} > \frac{\pi}{4}$.
Draw triangle OAB with right angle A, such that $AB = t$ and $OA = 1$. By the Pythagorean theorem, $OB = \sqrt{t^2+1}$. Then draw an altitude from A to OB at C. Triangles BOA and AOC are similar, so $OC = \frac{1}{\sqrt{t^2+1}}$ and $AC = \frac{t}{\sqrt{t^2+1}}$. Hence triangle AOC has area $\frac{1}{2} OC \cdot AC = \frac{t}{2(t^2+1)}$. Now draw a circle through A with center O. The sector of this circle intercepted by $\angle BOA$ has area $\frac{\arctan t}{2\pi} \cdot \pi r^2 = \frac{1}{2}\arctan t$, and contains triangle AOC. Therefore:

$$\left(\arctan \frac{1}{t}\right)(\arctan t) \ge \frac{\pi}{4}\arctan t > \frac{1}{2}\arctan t > \frac{t}{2(t^2+1)}.$$

Source: Bracken, Paul. "Problem 4467," *Crux Mathematicorum* **45**:7 (Sept. 2019), 414.