PROBLEM OF THE WEEK #6

(Fall 2021)

The Pioneers are trailing by one point in the final seconds. As the buzzer sounds, Maiah puts up a potential game-winner. Panicking, the defender slaps her wrist, causing the shot to spin out, but earning Maiah a pair of free throws. If she makes both, the Pioneers win! Making one out of two would force overtime, leaving each team the same chance of winning. There's also a third possibility, but let's not talk about that.

Given that the two free throws are independent, how much greater is the Pioneers' probability of winning if Maiah makes her first free throw than if she misses it?

Solution:

The difference between the two probabilities is 0.5.

Proof. Let p be the probability that Maiah makes any given free throw.

Assume that Maiah makes the first free throw. The Pioneers win if she makes her second free throw, with probability p. Even if she misses the second free throw, the Pioneers still have a 50% chance of winning in overtime. She misses with probability $1 - p$, so the total probability that the Pioneers win is $p + \frac{1}{2}(1 - p) = \frac{1}{2}p + \frac{1}{2}$.

Now assume that Maiah misses the first free throw. The Pioneers can only win in overtime, and to get to overtime, Maiah has to make her second free throw. So in this case, the probability that the Pioneers win is $\frac{1}{2}p$.

The difference between these probabilities is $\left(\frac{1}{2}p + \frac{1}{2}\right) - \frac{1}{2}p = \boxed{\frac{1}{2}}$. $\qquad\square$