University of Wisconsin
PLATTEVILLE
Department of Mathematics

Problem of the Week #4

(Fall 2021)

Aundra and her husband Dennis go to a get-together with four other couples. Each person there shakes hands with everyone he or she doesn't already know (and no one else). Later, Dennis surveys the nine other guests (besides himself) and learns that each of them shook hands with a different number of people.
How many people did Aundra shake hands with?

Solution:

Aundra shook hands with exactly four people.

Proof. Each guest knows their partner, so can shake hands with at most 8 people. The nine non-Dennis guests shook hands with different numbers of people, so their answers must have been the nine integers from 0 through 8. Name the non-Dennis guests $P_0, P_1, \ldots, P_8$ so that each P_i shook hands with exactly i people.
Now P_8 shook hands with everyone but their partner, and P_0 didn't shake hands with P_8, so P_0 and P_8 are partners.
Then P_7 shook hands with everyone but their partner and P_0, and P_1 only shook hands with P_8. Since P_1 didn't shake hands with P_7, and P_1 isn't P_0, we know that P_1 and P_7 are partners.
This means that P_6 shook hands with everyone but their partner, P_0, and P_1, and P_2 only shook hands with P_8 and P_7. Since P_2 didn't shake hands with P_6, and P_2 isn't P_0 or P_1, we see that P_2 and P_6 are partners.
Finally, P_5 shook hands with everyone but their partner, P_0, P_1, and P_2, while P_3 only shook hands with P_8, P_7, and P_6. Since P_3 didn't shake hands with P_5, and P_3 isn't P_0, P_1, or P_2, we conclude that P_3 and P_5 are partners.
But Aundra's partner is Dennis, who isn't P_i for any i. By elimination, Aundra is P_4. $\square$

Remark. To summarize, we showed that P_i shook hands with P_j if and only if $i + j \geq 9$, and that P_i and P_j are partners if and only if $i + j = 8$ (and $i \neq j$). By counting, we can learn that Dennis and Aundra shook hands with exactly the same people.

Source: Winkler, Peter. "Handshakes at a Party." *Mathematical Puzzles: A Connoisseur's Collection.* Wellesley: A K Peters (2004), 22, 26.