University of Wisconsin
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #3

(Fall 2021)

Find all real-valued functions F with domain $\mathbb{R}$ for which

$$F(u) - F(v) \leq (u - v)^2$$

for all real numbers u and v.

Solution:
The function F has the desired property if and only if F is constant.

Proof. If $F(x) = C$ for all x, then for all u and v,

$$F(u) - F(v) = C - C = 0 \leq (u - v)^2.$$

Conversely, suppose that F has the desired property. Let a be a real number.
For every real x, we are given both $F(x) - F(a) \leq (x - a)^2$ and $F(a) - F(x) \leq (a - x)^2$, so:

$$\begin{aligned}
|F(x) - F(a)| &\leq (x - a)^2 \\
|F(x) - F(a)| &\leq |x - a|^2 \\
\frac{|F(x) - F(a)|}{|x - a|} &\leq |x - a| \\
\left|\frac{F(x) - F(a)}{x - a}\right| &\leq |x - a|
\end{aligned}$$

That is, $-|x - a| \leq \dfrac{F(x) - F(a)}{x - a} \leq |x - a|$, and by the squeeze theorem, $\lim\limits_{x \to a} \dfrac{F(x) - F(a)}{x - a} = 0$.
Thus $F'(a) = 0$. Since this is true for all real a, we conclude that F is constant. □

Source: Khovanova, Tanya, and Alexey Radul. "Killer Problems." *The American Mathematical Monthly* **119**:10 (December 2012), 815–823.