Problem of the Week #2

(Fall 2021)

Find the sum of the numbers of the form $\frac{m}{n}$, where m and n are relatively prime positive divisors of 1000.

Solution:

If m and n are relatively prime positive divisors of 1000, then m and n have the form $2^i 5^j$, where $0 \le i, j \le 3$. Therefore $\frac{m}{n} = 2^a 5^b$, where $-3 \le a, b \le 3$, and the desired sum equals a product of two finite geometric series:

$$\begin{aligned}
& 2^{-3}5^{-3} + 2^{-3}5^{-2} + \cdots + 2^3 5^3 \\
&= \left(2^{-3} + \cdots + 2^2 + 2^3\right)\left(5^{-3} + \cdots + 5^2 + 5^3\right) \\
&= \frac{2^{-3}(1-2^7)}{1-2} \cdot \frac{5^{-3}(1-5^7)}{1-5} \\
&= \frac{127}{8} \cdot \frac{19531}{125} \\
&= \boxed{\frac{2480437}{1000}} \\
&= \boxed{2480.437}.
\end{aligned}$$

Source: Problem #11 of the 2000 American Invitational Mathematics Exam. In: Annin, Scott A., *A Gentle Introduction to the American Invitational Mathematics Exam*. Washington: The Mathematical Association of America (2015), 281.