University of Wisconsin
PLATTEVILLE
Department of Mathematics

Problem of the Week #1

(Fall 2021)

Given that a, b, c, and d are positive real numbers with $a \geq b \geq c \geq d > 0$, show that

$$(a + 2b + 3c + 4d)(a^2 + b^2 + c^2 + d^2) < (a + b + c + d)^3.$$

Solution:

Define: $\begin{cases} r &= a - b, \\ s &= b - c, \\ t &= c - d, \\ u &= d. \end{cases}$

By hypothesis, r, s, t, and u are non-negative, with $u > 0$.

We can check that: $\begin{cases} u &= d, \\ t + u &= c, \\ s + t + u &= b, \\ r + s + t + u &= a, \end{cases}$

and we can do a lot of manual labor or a bit of electronic computation to show that

$$\begin{aligned} &(a + b + c + d)^3 - (a + 2b + 3c + 4d)(a^2 + b^2 + c^2 + d^2) \\ =&(r + 2s + 3t + 4u)^3 - (r + 3s + 6t + 10u)[(r + s + t + u)^2 + (s + t + u)^2 + (t + u)^2 + u^2] \\ =&r^2s + r^2t + 4rs^2 + 14rst + 18rsu + 12rt^2 + 34rtu + 24ru^2 + 2s^3 + 12s^2t + 16s^2u + 21st^2 \\ &\quad + 62stu + 44su^2 + 9t^3 + 42t^2u + 60tu^2 + 24u^3. \end{aligned}$$

That quantity is positive, as desired, since all coefficients are positive, all variables are non-negative, and the final term $24u^3$ is strictly positive.

Source: Suggested by Problem 2 of the 2020 IMO, proposed by Stijn Cambie, Netherlands. Solution by Luke Robitaille, 10th grade, Robitaille Homeschool, Euless, TX. In: Bajnok, Béla, and Evan Chen, "Report on the 61st Annual International Mathematical Olympiad," *Mathematics Magazine* **94**:3 (June 2021), 215–224.