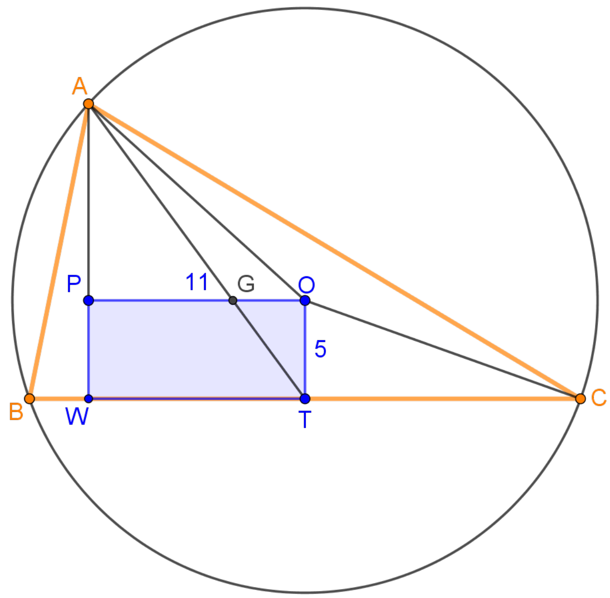
University of Wisconsin
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #10

(Fall 2020)

Let A, B, and C be three points on a circle with center O. Let T be the midpoint of BC, and let W be the foot of the altitude from A.
Suppose that the three altitudes intersect at P and $POTW$ is a rectangle with sides $OP = 11$ and $OT = 5$. Find the length of BC.



Solution:

$BC = 28$.

Proof. Let G be the centroid of $\triangle ABC$. Since O and P are the circumcenter and orthocenter of $\triangle ABC$ respectively, the points P, G, and O are on a single line (the triangle's "Euler line"). We know that G lies two-thirds of the way from A to T. By the similarity of $\triangle APG$ and $\triangle AWT$, P lies two-thirds of the way from A to W. Therefore $AP = 10$.
Now the Pythagorean theorem tells us that $AO = \sqrt{10^2 + 11^2} = \sqrt{221}$. AO and OC are radii of the circle, so $OC = \sqrt{221}$. We can use the Pythagorean theorem again to find that $CT = \sqrt{221 - 25} = 14$, and so $BC = 28$. $\square$

Source: Problem A-1 of the 58th William Lowell Putnam Mathematical Competition (1997).