UNIVERSITY OF WISCONSIN
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

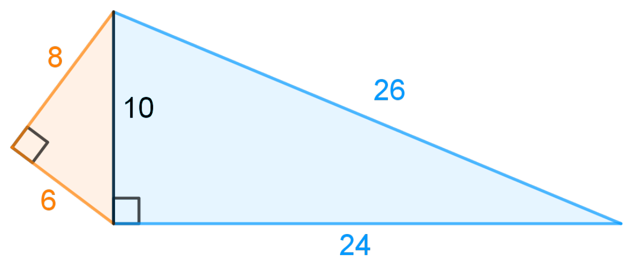
Problem of the Week #9

(Fall 2020)

Find all right triangles that have a side of length 10 and two other sides with integer lengths.

Solution:

The side lengths of such a triangle are either $\{6, 8, 10\}$ or $\{10, 24, 26\}$.



Proof. Let T be a right triangle with integer sides, including a side of length 10, and let d be the greatest common divisor of the side lengths. Then $d \mid 10$, so $d \in \{1, 2, 5, 10\}$.

First, we consider the case $d = 1$. Then T is called a *primitive* triangle. Euclid showed that in this case there are relatively prime positive integers s and t, one odd and the other even, for which the side lengths are $a = s^2 - t^2$, $b = 2st$, and $c = s^2 + t^2$. Note that b is a multiple of 4, while a and c are odd. Thus no side of a primitive triangle can have length 10.

If $d \in \{5, 10\}$, then scaling T down by a factor of d produces a primitive triangle with a side of length 1 or 2. But again we have $a = s^2 - t^2$, $b = 2st$, and $c = s^2 + t^2$, so $a \geq 3$, $4 \mid b$, and $c \geq 5$ — a contradiction.

Finally, if $d = 2$, then scaling T down by a factor of d produces a primitive triangle with a side of length 5. Since $4 \mid b$, we know $b \neq 5$. If $5 = a = s^2 - t^2 = (s + t)(s - t)$, then $s + t = 5$ and $s - t = 1$, so $(s, t) = (3, 2)$, $(a, b, c) = (5, 12, 13)$, and T has side lengths $\{10, 24, 26\}$. If $5 = c = s^2 + t^2 \geq 2^2 + 1^2 = 5$, we have $(s, t) = (2, 1)$, so $(a, b, c) = (3, 4, 5)$, and T has side lengths $\{6, 8, 10\}$. $\square$

Source: John J. Watkins, "T^2 TEN T^2: Ten Terribly Tempting Elementary Number Theory Tidbits," in *G4G10 Exchange Book* (`https://www.gathering4gardner.org/g4g10gift/G4G10-Book-Volume2-ForWeb.pdf`).