# University of Wisconsin Platteville
## Department of Mathematics

# Problem of the Week #8
(Fall 2020)

Let $\{a_1, a_2, \dots\}$ be a strictly increasing sequence of positive integers: if $m > n$, then $a_m > a_n$. Assuming that $a_{(a_n)} = 3n$ for every positive integer $n$, find $a_{1000}$.

**Solution:**

$a_{1000} = 1729.$*

*Proof.* I claim that when $k$ is a non-negative integer, we have $a_{3^k} = 2 \cdot 3^k$ and $a_{2\cdot 3^k} = 3^{k+1}$. The proof is by induction on $k$. Suppose for the sake of contradiction that $a_1 = 1$. Then $a_1 = a_{a_1} = 3$, which is impossible. So $a_1 \geq 2$. Hence $3 = a_{a_1} \geq a_2$. On the other hand, $a_2 > a_1 \geq 2$, so because $a_2$ is an integer, $a_2 \geq 3$. The only possibility is $a_2 = 3$. This means $a_{a_1} = 3 = a_2$, and so $a_1 = 2$. This proves the claim for $k = 0$.
Now assume for induction that $a_{3^k} = 2 \cdot 3^k$ and $a_{2\cdot 3^k} = 3^{k+1}$ for some $k \geq 0$. Hence

$$a_{3^{k+1}} = a_{a_{2\cdot 3^k}} = 3(2 \cdot 3^k) = 2 \cdot 3^{k+1}$$

and

$$a_{2\cdot 3^{k+1}} = a_{a_{3^{k+1}}} = 3(3^{k+1}) = 3^{(k+1)+1},$$

completing the induction.
Finally, observe that for any $n$, $a_{n+1} > a_n$, and since both terms are integers, $a_{n+1} - a_n \geq 1$. Adding several of these equations yields a telescoping sum: $a_{n+y} - a_n \geq y$. By substitution, if $m > n$, then $a_m - a_n \geq m - n$. In particular, if $3^k < t < 2 \cdot 3^k$, then $\begin{cases} a_t - a_{3^k} & \geq & t - 3^k, \\ a_{2\cdot 3^k} - a_t & \geq & 2 \cdot 3^k - t. \end{cases}$
We can solve both equations for $a_t$ to obtain:

$$\begin{cases} a_t & \geq & t - 3^k + a_{3^k} & = & t - 3^k + 2 \cdot 3^k & = & t + 3^k, \\ a_t & \leq & t - 2 \cdot 3^k + a_{2\cdot 3^k} & = & t - 2 \cdot 3^k + 3^{k+1} & = & t + 3^k. \end{cases}$$

Since 1000 is between $3^6 = 729$ and $2 \cdot 3^6 = 1458$, we obtain $a_{1000} = 1000 + 3^6 = 1729$. □

*Remark.* It's not relevant to our specific question, but we can find the remaining terms as well. If $2 \cdot 3^k < t < 3^{k+1}$, then let $s = t - 3^k$. Since $3^k < s < 2 \cdot 3^k$, we have $a_s = s + 3^k = t$, and so $a_t = a_{a_s} = 3s = 3(t - 3^k)$. For instance, since 2020 is between $2 \cdot 3^6 = 1458$ and $3^7 = 2187$, we get $a_{2020} = 3(2020 - 729) = 3873$.

**Source:** Velleman, Daniel J., and Stan Wagon. *Bicycle or Unicycle?* Providence: MAA Press (2020), 17.

*This is a very interesting number, because it is the smallest integer that can be expressed as a sum of positive cubes in more than one way: $12^3 + 1^3 = 1728 + 1$ and $10^3 + 9^3 = 1000 + 729$.