University of Wisconsin
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #7

(Fall 2020)

A **chunk** of an integer n is a sequence of zero or more consecutive digits taken from the base-10 representation of n. For example, the chunks of 4216847 include 168, 84, 4216847, and 1, but not 9 or 4187.

Prove that if m and n are integers with at least 10 digits, then m and n contain chunks (not both empty) whose digits have the same sum.

Solution:

Proof. Let $m_1, \dots, m_{10}$ and $n_1, \dots, n_{10}$ be the first 10 digits of m and n respectively.
For each k with $1 \le k \le 10$, let $a_k = \sum_{i=1}^{k} m_i$ and $b_k = \sum_{i=1}^{k} n_i$. If $a_{10} = b_{10}$, we are done. Otherwise, suppose without loss of generality that $b_{10} > a_{10}$. For each k, let $t(k)$ be the least index for which $b_{t(k)} \ge a_k$. Note that $b_{t(k)-1} < a_k$. Notice also that if $k < \ell$, then $b_{t(\ell)} \ge a_\ell \ge a_k$, so $t(k) \le t(\ell)$, by the minimality of $t(k)$.
If there is any k for which $b_{t(k)} = a_k$, we are done. Otherwise, $b_{t(k)} > a_k$ for all k. Let $c_k = b_{t(k)} - a_k$. By hypothesis, $c_k > 0$. On the other hand,

$$c_k = b_{t(k)} - a_k < b_{t(k)} - b_{t(k)-1} = n_{t(k)} \le 9.$$

There are ten terms in $\{c_1, \dots, c_{10}\}$, and their values are in $\{1, \dots, 9\}$, so by the pigeonhole principle, there must be distinct indices $p < q$ with

$$\begin{aligned} c_p &= c_q \\ b_{t(p)} - a_p &= b_{t(q)} - a_q \\ a_q - a_p &= b_{t(q)} - b_{t(p)} \\ \sum_{i=p+1}^{q} m_i &= \sum_{i=t(p)+1}^{t(q)} n_i \end{aligned}$$

Since $p < q$, there's at least one term in $\sum_{i=p+1}^{q} m_i$. $\square$

Source: Winkler, Peter. "Red and Blue Dice," in *Mathematical Mind-Benders*. Wellesley: A K Peters, Ltd. (2007), 23, 33-34.