University of Wisconsin Platteville
Department of Mathematics

Problem of the Week #6
(Fall 2020)

Given a real number x, find all n-tuples $(y_1, \ldots, y_n)$ that solve the real equation

$$\sum_{k=1}^{n} \sqrt{x - y_k} = n + \sqrt{(1-x)n + \sum_{k=1}^{n} y_k}.$$

Solution:
The equation holds only when $y_1 = \cdots = y_n = x - 1$.

Proof. Let $z_k = \sqrt{x - y_k}$, which is assumed to be real. So $y_k = x - z_k^2$, and in that notation, the given equation is

$$\begin{aligned} \sum_{k=1}^{n} z_k &= n + \sqrt{(1-x)n + \sum_{k=1}^{n}(x - z_k^2)} \\ -n + \sum_{k=1}^{n} z_k &= \sqrt{(1-x)n + nx - \sum_{k=1}^{n} z_k^2} \\ -n + \sum_{k=1}^{n} z_k &= \sqrt{n - \sum_{k=1}^{n} z_k^2} \end{aligned}$$

The left side of this equation is real, so the right side is real, and therefore non-negative. That means the left side is also non-negative. Hence $\sum_{k=1}^{n} z_k \geq n$ and $\sum_{k=1}^{n} z_k^2 \leq n$. But then:

$$\begin{aligned} \left(\sum_{k=1}^{n} z_k^2\right) - 2\left(\sum_{k=1}^{n} z_k\right) &\leq n - 2n \\ \sum_{k=1}^{n}(z_k^2 - 2z_k) &\leq -n \\ \sum_{k=1}^{n}(z_k^2 - 2z_k + 1) &\leq 0 \\ \sum_{k=1}^{n}(z_k - 1)^2 &\leq 0 \end{aligned}$$

It follows that each $z_k = 1$, and therefore each $y_k = x - 1$. □

Source: Sokolovsky, Yevgeniy (solution by Missouri State University problem solving group). “Additive Roots.” *Math Horizons* **27**:4 (April 2020), 31.