UNIVERSITY OF WISCONSIN
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #5

(Fall 2020)

Find a triangle with area 168, whose side lengths are integers, with all three vertices lying on a circle whose radius is a perfect square.

Solution:

A triangle with sides of length 14, 30, and 40 has area 168 and circumradius 25.

Let $a \le b \le c$ be the side lengths of a triangle T, with opposite angles A, B, and C as usual. By the law of sines, there is some q with $a = q\sin A$, $b = q\sin B$, and $c = q\sin C$. The area of T is $\frac{1}{2}ab\sin C = 168$, so $abc = 336q$. Then the circumradius of T is $\frac{abc}{4(168)} = \frac{q}{2}$, so there is some integer k with $q = 2k^2$. Since $\sin C \le 1$, we know that $c \le q \le 2k^2$, and also that $2 \cdot 168 \cdot q = abc \le q^3$, which means $q \ge \sqrt{336}$. Therefore $k^2 \ge \sqrt{84} > 9$, so $k > 3$.

By Heron's formula,

$$168 = \sqrt{\frac{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}{16}}.$$

We can rule out any (a,b,c) with exactly one odd entry, since then the numerator in Heron's formula, $(a+b+c)(a+b-c)(a-b+c)(-a+b+c)$, would be odd.

There are no solutions with $k = 4$, which would yield $abc = Nk^2 = 2^9 \cdot 3 \cdot 7 = 10752$. Hence $c \mid 10752$ and $c \ge \sqrt[3]{abc} = \sqrt[3]{10752} > 22$, as well as $c \le 2k^2 = 32$. The remaining possibilities are shown in the table at right, but none of them has area $168 = \sqrt{28224}$.

a	b	c	area
16	24	28	$\sqrt{36720}$
12	28	32	$\sqrt{27648}$
14	24	32	$\sqrt{24255}$

However, with $k = 5$, $abc = Nk^2 = 2^5 \cdot 3 \cdot 5^2 \cdot 7 = 16800$. We know that $c \mid 16800$, that $c \ge \sqrt[3]{abc} = \sqrt[3]{16800} > 25$, and that $c \le 2k^2 = 50$. The triangle inequality implies $a + b > c$. The remaining possibilities are shown in the table at right.

a	b	c	area
20	28	30	$\sqrt{73359}$
21	25	32	$\sqrt{68796}$
15	32	35	$\sqrt{57564}$
14	30	40	$\sqrt{28224}$
10	40	42	$\sqrt{39744}$

We have found $(a,b,c) = (14,30,40)$, with area 168 and circumradius 25. This triangle can be realized with its vertices at $A = (0,0)$, $B = (40,0)$, and $C = (28.8, 8.4)$; the circumcenter is $(20,-15)$. This is the smallest triangle, by area, with integer side lengths, integer area, and a circumradius that is a perfect square.