University of Wisconsin
PLATTEVILLE
Department of Mathematics

Problem of the Week #4

(Fall 2020)

Evaluate $\int_0^\infty \left(x - \frac{x^3}{2} + \frac{x^5}{2\cdot 4} - \frac{x^7}{2\cdot 4\cdot 6} + \dots\right)\left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} + \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \dots\right) dx.$

Solution:

First, $e^x = \sum_{k=0}^\infty \frac{x^k}{k!}$, so $xe^{-x^2/2} = \sum_{k=0}^\infty \frac{(-1)^k x^{2k+1}}{2^k\cdot k!} = x - \frac{x^3}{2} + \frac{x^5}{2\cdot 4} - \frac{x^7}{2\cdot 4\cdot 6} + \dots.$
Therefore, our integral is $\int_0^\infty xe^{-x^2/2}\left[\sum_{k=0}^\infty \frac{x^{2k}}{(2^k\cdot k!)^2}\right] dx = \sum_{k=0}^\infty \frac{1}{(2^k\cdot k!)^2}\left[\int_0^\infty x^{2k+1}e^{-x^2/2}\,dx\right].$
I claim that $\int_0^\infty x^{2k+1}e^{-x^2/2}\,dx = 2^k\cdot k!$. The proof is by induction on k. Taking $u = -x^2/2$,

$$\int_0^\infty xe^{-x^2/2}\,dx = \lim_{a\to\infty}\int_0^a xe^{-x^2/2}\,dx = \lim_{a\to\infty}\int_0^{-a^2/2} -e^u\,du = \lim_{a\to\infty} 1 - e^{-a^2/2} = 1 = 2^0\cdot 0!,$$

so the claim holds when $k = 0$. Suppose now that $m \ge 0$ and $\int_0^\infty x^{2m+1}e^{-x^2/2}\,dx = 2^m\cdot m!$. Apply integration by parts with $u = x^{2m+2}$ and $v = -e^{-x^2/2}$ to obtain

$$\begin{aligned}\int_0^\infty x^{2m+3}e^{-x^2/2}\,dx &= \lim_{a\to\infty} -x^{2m+2}e^{-x^2/2}\Big|_0^a + \int_0^a (2m+2)x^{2m+1}e^{-x^2/2}\,dx \\ &= \lim_{a\to\infty} -a^{2m+2}e^{-a^2/2} + (2m+2)\left(2^m\cdot m!\right) \\ &= 2^{m+1}\cdot(m+1)!,\end{aligned}$$

completing the induction. Thus our integral equals

$$\sum_{k=0}^\infty \frac{1}{(2^k\cdot k!)^2}\left[2^k\cdot k!\right] = \sum_{k=0}^\infty \frac{1}{2^k\cdot k!} = \sum_{k=0}^\infty \frac{(1/2)^k}{k!} = \boxed{\sqrt{e}}.$$

Source: Problem A-3 of the 58th William Lowell Putnam Mathematical Competition (1997).