



PROBLEM OF THE WEEK #3
(Fall 2020)

My ultra-secret society urgently needs to deliver a vital piece of new technology to our hidden co-conspirator, Master Agent Z, without letting it fall into the wrong hands.

We've designed the item to be composed of five sophisticated and secret components, which are useless on their own. In fact, even if our enemies should capture four of the five components, it wouldn't do them any good at all. Unfortunately, the same is true for Master Agent Z — we have to make sure she receives all five components.

Worse, we know that two enemy spies will be on the lookout — but we have no idea where they are and can't predict their movements. Several of us have volunteered to carry the components to Master Agent Z, traveling at the same time by different and circuitous routes, but we have to be prepared for the possibility that two of the couriers will be captured. We've produced several copies of each component, and the couriers can each carry multiple components if necessary.

What is the smallest number of couriers we can send to ensure that at least one copy of each component gets to Master Agent Z, but without giving the enemy spies a chance to obtain copies of all five components by capturing two couriers?

Solution:

We must send seven couriers.

Proof: Let's label the components A, B, C, D, and E. Seven couriers can do the job, carrying the components shown in the table. If courier 1 is captured, then at least one of {2,3}, of {4,5}, and of {6,7} will get to Master Agent Z, and those three together will have all five components. Also, the other captured courier (if any) won't have both D and E, so the enemies won't get all five components. On the other hand, if courier 1 is not captured, then at least one of {2,3,4} and at least one of {5,6,7} will get through, and those two, together with 1, will have all five components. Also, captured couriers will have two components each, so the enemies will get at most four components.

1	{A,B,C}
2	{C,D}
3	{C,D}
4	{B,D}
5	{B,E}
6	{A,E}
7	{A,E}

Notice, however, that each component has to be sent with at least three couriers, so that at least one copy of that component will get to Master Agent Z. Therefore:

- No courier x can carry 4 components: capturing x and a courier with the other component would give the enemies every component.
- If no courier carries more than two components, then there must be at least eight couriers to carry the $3 \times 5 = 15$ components that must be carried.
- If any courier x carries exactly three components (say A, B, and C), then no courier y can carry both D and E, or capturing x and y would give the enemies all five components. So there must be at least seven couriers: three carrying D, three others carrying E, and x . \square