



PROBLEM OF THE WEEK #2
(Fall 2020)

Find every function $f(x)$ with the property that $f(x) - 3f(1-x) = x^2$ for every real x .

Solution:

The only such function is $f(x) = \frac{-3 + 6x - 4x^2}{8}$.

Proof. By substituting $1-x$ for x in the given equation, we get

$$\begin{aligned} f(1-x) - 3f(1-(1-x)) &= (1-x)^2 \\ f(1-x) - 3f(x) &= 1-2x+x^2 \\ 3f(1-x) - 9f(x) &= 3-6x+3x^2 \\ f(x) - 3f(1-x) + 3f(1-x) - 9f(x) &= x^2 + 3 - 6x + 3x^2 \\ -8f(x) &= 3 - 6x + 4x^2 \\ f(x) &= \boxed{\frac{-3 + 6x - 4x^2}{8}}. \end{aligned}$$

Conversely,

$$\begin{aligned} &\frac{-3 + 6x - 4x^2}{8} - 3\left(\frac{-3 + 6(1-x) - 4(1-x)^2}{8}\right) \\ &= \frac{-3 + 6x - 4x^2}{8} + \frac{9 - 18(1-x) + 12(1-2x+x^2)}{8} \\ &= \frac{-3 + 6x - 4x^2 + 9 - 18 + 18x + 12 - 24x + 12x^2}{8} \\ &= \frac{8x^2}{8} \\ &= x^2. \end{aligned}$$

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