University of Wisconsin
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Department of Mathematics

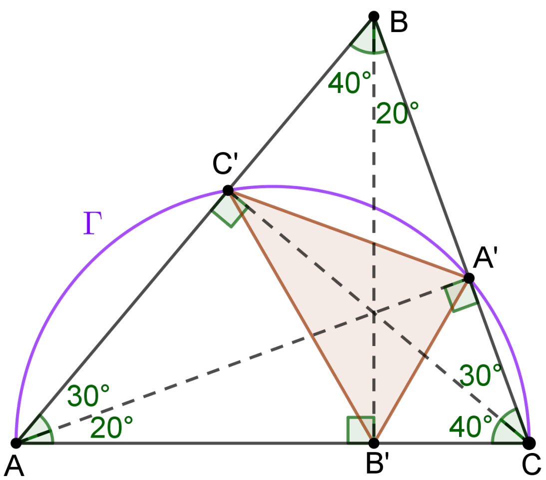
Problem of the Week #10

(Fall 2019)

Given a triangle T with angle measures $50°$, $60°$, and $70°$, what are the angle measures of the triangle whose vertices are the feet of the altitudes of T?

Solution:

The angle measures are $40°$, $60°$, and $80°$.



Proof. Suppose $\triangle ABC$ has $\angle A = 50°$, $\angle B = 60°$, and $\angle C = 70°$. Let A' (resp. B', C') be the foot of the altitude from A (resp. B, C), as shown in the figure. It is easy to use the triangles with right angles at A', B', and C' to find the measures of all angles with vertices at A, B, and C.

Draw a circle Γ with diameter AC. Because $\angle AA'C$ and $\angle AC'C$ are right angles that intercept the diameter of Γ, the points A' and C' lie on Γ. Now $\angle A'C'C = \angle A'AC = 20°$, because they intercept the same arc of Γ, and likewise $\angle AA'C' = \angle ACC' = 40°$.

Drawing circles on the other two sides of $\triangle ABC$ and using the same argument, we get:

- $\angle AA'B' = \angle ABB' = 40°$,
- $\angle BB'A' = \angle BAA' = 30°$,
- $\angle BB'C' = \angle BCC' = 30°$,
- $\angle CC'B' = \angle CBB' = 20°$.

Thus the angle measures of $\triangle A'B'C'$ are $\angle A' = 40° + 40° = 80°$, $\angle B' = 30° + 30° = 60°$, and $\angle C' = 20° + 20° = 40°$. $\square$

Source: Suggested by MAA American Mathematics Competitions, “Friday’s Problem of the Day,” MathFest 2019.