University of Wisconsin
Platteville
Department of Mathematics

Problem of the Week #8

(Fall 2019)

Find all positive integers b for which $2^b - 1$ is a divisor of $2^{2019} - 1$.

Solution:
The solutions are the positive divisors of 2019: specifically, $b \in \{1, 3, 673, 2019\}$.

Proof. We will show, more generally, that $2^b - 1$ divides $2^a - 1$ exactly when b divides a. Let q and r be integers for which $a = qb + r$, with $0 \leq r < b$. [That is, when you divide a by b, the quotient is q and the remainder is r.] Let $x = 2^b$, so $x^q = (2^b)^q = 2^{qb}$. Then

$$2^{qb} - 1 = x^q - 1 = (x - 1)(x^{q-1} + x^{q-2} + \cdots + x + 1) = (2^b - 1)(2^{b(q-1)} + 2^{b(q-2)} + \cdots + 2^b + 1).$$

Multiply both sides of this equation by 2^r:

$$\begin{aligned}
2^{qb+r} - 2^r &= (2^b - 1)(2^{b(q-1)+r} + 2^{b(q-2)+r} + \cdots + 2^{b+r} + 2^r) \\
2^{qb+r} - 2^r + 2^r - 1 &= (2^b - 1)(2^{b(q-1)+r} + 2^{b(q-2)+r} + \cdots + 2^{b+r} + 2^r) + 2^r - 1 \\
2^a - 1 &= (2^b - 1)(2^{b(q-1)+r} + 2^{b(q-2)+r} + \cdots + 2^{b+r} + 2^r) + (2^r - 1)
\end{aligned}$$

Now, because $0 \leq r < b$, we know $1 \leq 2^r < 2^b$, so $0 \leq 2^r - 1 < 2^b - 1$. Thus $2^r - 1$ is the remainder when we divide $2^a - 1$ by $2^b - 1$. So $2^b - 1$ is a divisor of $2^a - 1$ exactly when $2^r - 1 = 0$, which happens only when $2^r = 1$ — but this means that $r = 0$ and therefore a is divisible by b. $\square$

Source: Suggested by Problem B3 of the Seventy-Ninth William Lowell Putnam Mathematical Competition (2018). A solution by Fiona Bradley (Northwestern University) appears in *American Mathematical Monthly* **126**:8 (October 2019), p. 684.