University of Wisconsin
PLATTEVILLE
Department of Mathematics

Problem of the Week #5

(Fall 2019)

You are about to play a game. You will roll an even number of fair 20-sided dice (with their faces numbered from 1 to 20 as usual), and you win a prize if more than half of them come up showing single-digit numbers. If you get to choose, how many dice should you roll?

Solution:

You should roll 10 dice.

Proof. On any given die, the probability of a success (a single-digit roll) is $p = \frac{9}{20}$, and $q = 1 - p = \frac{11}{20}$ is the probability of a failure.
Suppose you choose to roll $2k+2$ dice. You can start by rolling $2k$ dice, then finish by rolling the other two. Once you've rolled the first $2k$ dice, you can award yourself a "virtual" win or loss, according to whether you would have won the prize or not if you'd chosen to roll only $2k$ dice. Usually, your result is the same as your virtual result. The exceptions are:

- You roll k successes on the first $2k$ dice (a virtual loss), then roll two successes.
- You roll $k+1$ successes on the first $2k$ dice (a virtual win), then roll two failures.

Therefore, by choosing to roll $2k+2$ dice instead of $2k$ you have increased your probability of winning the prize by

$$\Delta(k) = p^2\binom{2k}{k}p^k q^k - q^2\binom{2k}{k+1}p^{k+1}q^{k-1}.$$

Your k is locally optimal when $\Delta(k) \geq 0$ but $\Delta(k+1) \leq 0$, and $\Delta(k) \geq 0$ if and only if:

$$\begin{aligned}
p^2\binom{2k}{k}p^k q^k &\geq q^2\binom{2k}{k+1}p^{k+1}q^{k-1} \\
\frac{(2k)!}{k!k!}p^{k+2}q^k &\geq \frac{(2k)!}{(k+1)!(k-1)!}p^{k+1}q^{k+1} \\
\frac{p}{k} &\geq \frac{q}{k+1} \\
kp + p &\geq kq \\
p &\geq k(q-p) \\
k &\leq \frac{p}{q-p} = \frac{9/20}{2/20} = \frac{9}{2}.
\end{aligned}$$

So the best choice is to roll $2k+2$ dice for $k = 4$; *i.e.,* you should roll 10 dice. □

Source: "Winning an Unfair Game." Frederick Mosteller, *Fifty Challenging Problems in Probability with Solutions*. Reading: Addison-Wesley (1965), p. 11, 66–67.