University of Wisconsin
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Problem of the Week #4

(Fall 2019)

How many ordered pairs of integers (a, b) with $2 \leq a \leq 2019$ and $2 \leq b \leq 2019$ have the property that $\log_a b + 6 \log_b a = 5$?

Solution:

Let $x = \log_b a$. (Since $a \neq 1$, we have $x \neq 0$.) By the base change formula,

$$\log_a b = \frac{\log_b b}{\log_b a} = \frac{1}{x},$$

so we are looking for pairs (a, b) with:

$$\begin{aligned} \tfrac{1}{x} + 6x &= 5 \\ 6x^2 - 5x + 1 &= 0 \\ (3x-1)(2x-1) &= 0 \\ x = 1/3 \quad \text{or} \quad & x = 1/2 \end{aligned}$$

If $\log_b a = x = \frac{1}{3}$, then $a = b^{1/3} \leq 2019^{1/3} < 13$, so $2 \leq a \leq 12$. Each of these values of a can be paired with the integer $b = a^3 \leq 2019$.
If $\log_b a = x = \frac{1}{2}$, then $a = b^{1/2} \leq 2019^{1/2} < 45$, so $2 \leq a \leq 44$. Each of these values of a can be paired with the integer $b = a^2 \leq 2019$.
This yields a total of $11 + 43 = \boxed{54}$ ordered pairs.

Source: 2005 AIME-2, Problem #5. In Scott A. Annin, *A Gentle Introduction to the American Invitational Mathematics Exam.* MAA Press, 2015, pp. 123, 302.