University of Wisconsin
PLATTEVILLE
Department of Mathematics

Problem of the Week #3

(Fall 2019)

The number 142857 becomes exactly three times as great when its initial (decimal) digit is transferred from its beginning to its end:

$$142857 \times 3 = 428571.$$

Show that there is no positive integer which becomes exactly seven times as great when its initial (decimal) digit is transferred from its beginning to its end.

Solution:

Let n be a positive integer with $k+1$ digits. Then $n = d \cdot 10^k + r$, where d is the initial digit of n and r is a number with (up to) k digits.

When the digit d moves from the beginning to the end of the number, the new number is $10r + d$, which still has (up to) $k+1$ digits.

Suppose for the sake of contradiction that $10r+d = 7n$. Since $7n$ is a $(k+1)$-digit number, the digit d must be 1; otherwise $7n \geq 14 \cdot 10^k$ would have $k+2$ digits. Now we have $10r+1 = 7n$. Solving for r, we find that

$$r = \frac{7 \cdot 10^k - 1}{3} > \frac{7}{3} \cdot 10^k > 10^k.$$

This means that r has more than k digits, which is the desired contradiction.

Source: D.O. Shklarsky, *et al. The USSR Olympiad Problem Book*. Mineola: Dover Publications, Inc, 1993. pp. 12, 111.