Problem of the Week #2

(Fall 2019)

How many 19-digit strings of Xs and Os are there that begin with an O, end with an O, contain no two consecutive Os, and contain no three consecutive Xs?

Solution:

There are 65 such strings.

Proof. Suppose such a string contains k Os, and therefore $19-k$ Xs. The Os separate the Xs into $k-1$ "bunches." Since there are no two consecutive Os, each bunch contains at least one X; since there are no three consecutive Xs, each bunch contains at most two Xs. Thus each bunch must contain a "mandatory X," and may also contain an "extra X."

There are $k-1$ mandatory Xs, and there can't be more mandatory Xs than there are Xs, so $k-1 \leq 19-k$, which means $k \leq 10$. The other $20-2k$ Xs are extra Xs, and there can't be more extra Xs than there are bunches, so $20-2k \leq k-1$, which means $k \geq 7$.

The number of strings with k Os is now equal to the number of ways to choose $20-2k$ of the $k-1$ bunches to contain our extra Xs. Therefore, the total number of strings of the desired type is

$$\sum_{k=7}^{10} \binom{k-1}{20-2k} = \binom{6}{6} + \binom{7}{4} + \binom{8}{2} + \binom{9}{0} = 1 + 35 + 28 + 1 = 65.$$

□

Source: MAA American Mathematics Competitions, "Wednesday's Problem of the Day," MathFest 2019.