UNIVERSITY OF WISCONSIN
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #10

(Fall 2018)

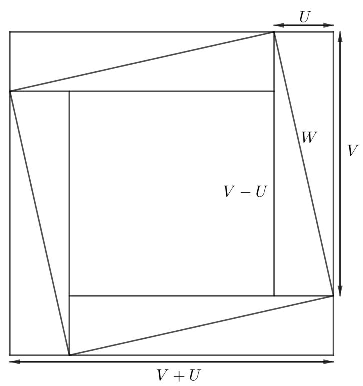
Find a rational number y for which $\sqrt{y^2-5}$ and $\sqrt{y^2+5}$ are both rational.

Solution:

If k is a common denominator of y, $\sqrt{y^2-5}$, and $\sqrt{y^2+5}$, then $P=ky$, $Q=k\sqrt{y^2-5}$, and $R=k\sqrt{y^2+5}$ are integers, and so

$$\left(Q^2,P^2,R^2\right)=\left(P^2-5k^2,P^2,P^2+5k^2\right)$$

are three perfect squares in arithmetic progression.



On the other hand, if the right triangles in the figure have side lengths $U<V<W$ and area A, then

$$\left((V-U)^2,W^2,(V+U)^2\right)=\left(W^2-4A,W^2,W^2+4A\right)$$

are three perfect squares in arithmetic progression.

Hence we shall seek $ky=P=W$, where (U,V,W) is a (primitive) Pythagorean triple, $A=\frac{1}{2}UV$, and $4A=5k^2$. Because 2 and 5 are prime, this last equation implies that k must be even and therefore $A/5=(k/2)^2$ must be a perfect square. A short search yields:

$$(U,V,W)=(9,40,41)\Rightarrow A=180\Rightarrow(k/2)^2=36\Rightarrow|k|=12\Rightarrow|y|=\frac{41}{12}.$$

Taking $\boxed{y=\pm\frac{41}{12}}$, we have: $\sqrt{y^2-5}=\sqrt{\frac{1681-720}{144}}=\frac{31}{12}$ and $\sqrt{y^2+5}=\sqrt{\frac{1681+720}{144}}=\frac{49}{12}$.

Remark. According to [Che18], this problem was proposed by John of Palermo in 1220.

Source:

[Che18] Stephanie Chen, *Rational right triangles of a given area*, American Mathematical Monthly **125** (August 2018), no. 8, 689-703.