UNIVERSITY OF WISCONSIN
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #7

(Fall 2018)

An even number of students took a quiz. Every student's score was an integer, every integer from the lowest score to the highest occurred at least once, and the average score was exactly 8.2. Show that the students can be split into two groups of the same size with the same average score.

Solution:

Form a list L by sorting the $2N$ quiz scores in decreasing order. Let A and B be empty lists. Let S_A and S_B be the sums of the entries in A and B respectively (so initially $S_A = 0 = S_B$). Now do the following:

1. Let $u \geq v$ be the first two entries in L. Note that $u - v \in \{0, 1\}$, because every integer from the lowest score to the highest occurs in L at least once.

2. If $S_A - S_B = 0$, then add u to A and add v to B. Otherwise, $S_A - S_B = 1$; in this case, add v to A and u to B. Either way, remove u and v from L.

3. Observe that L still has an even number of entries, A and B still have the same number of entries, and $S_A - S_B$ still equals either 0 or 1.

4. If L is not empty, then return to step 1.

At the end of this process, half of the students' scores are in A, the other half are in B, and $S_A - S_B \in \{0, 1\}$. However, $S_A + S_B = 2N(8.2) = 82N/5$. Since 82 is even and $\gcd(2, 5) = 1$, this means that $S_A + S_B$ is even. Therefore, $S_A - S_B$ is also even, and so $S_A - S_B = 0$, making the averages of the scores in A and B equal, as desired.

Remark. This is problem A4 of the 2017 Putnam exam, and [KU18] attributes this solution to student Laura Pierson.

Source:

[KU18] Mark Krusemeyer and Daniel H. Ullman, *The seventy-eighth William Lowell Putnam Mathematical Competition*, American Mathematical Monthly **125** (August 2018), no. 8, 675-688.