University of Wisconsin
PLATTEVILLE
Department of Mathematics

Problem of the Week #2

(Fall 2018)

This weekend I went to a 20-team round robin Quidditch tournament, which means that each team played every other team exactly once.
At the end of the tournament, each team's number of wins was a perfect square, and at least as many teams finished with 16 wins as with 9 wins. There were no ties.
Exactly how many teams ended up with 16 wins?

Solution:
Each team played exactly 19 games. Let (a,b,c,d,e) be the number of teams with $(0,1,4,9,16)$ wins respectively. We are given that $d \le e$.
The 16-win teams lost 3 games each, for a total of $3e$ losses. There were $\binom{e}{2}$ games between 16-win teams. Thus

$$\begin{aligned} \binom{e}{2} &\le 3e \\ e(e-1) &\le 6e \\ e^2 - 7e &\le 0 \\ e &\le 7 \end{aligned}$$

On the other hand: There were $\binom{20}{2} = 190$ games. Thus

$$\begin{cases} a+b+c+d+e = 20, & (1) \\ b+4c+9d+16e = 190. & (2) \end{cases}$$

Subtracting (1) from (2) yields $-a+3c+8d+15e = 170$. Solving this equation, and (1), for c, and equating the results, we obtain:

$$\begin{aligned} \frac{1}{3}(170+a-8d-15e) &= 20-a-b-d-e \\ \frac{1}{3}(110+4a+3b) &= 4e+\frac{5}{3}d \\ \frac{1}{3}(110) &\le \frac{17}{3}e \qquad \text{(since } a \ge 0,\ b \ge 0, \text{ and } d \le e) \\ 6.47 &< e \end{aligned}$$

Since $6 < e \le 7$ and e is an integer, we have $\boxed{e = 7}$.

Source: Inspired by Velleman, Dan, and Rob Leduc. "Ultimate Round Robin." Macalester College Problem of the Week 886 (31 March 1999), available at `http://stanwagon.com/potw/spring99/p886.html`.