University of Wisconsin
PLATTEVILLE
Department of Mathematics

Problem of the Week #9

(Fall 2017)

Find every non-negative integer n for which $n! - 1$ is a perfect square.

Solution:
It is a famous fact that when a perfect square is divided by 3, the remainder is always 0 or 1:

- $(3k)^2 = 9k^2 = 3(3k^2)$.
- $(3k+1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$.
- $(3k+2)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$.

But if $n \geq 3$, then $n!$ is a multiple of 3. That means that when $n! - 1$ is divided by 3, the remainder is 2, so $n! - 1$ is not a perfect square.
But $n! - 1$ is a perfect square for every other possible value of n:

n	$n! - 1$
0	0
1	0
2	1