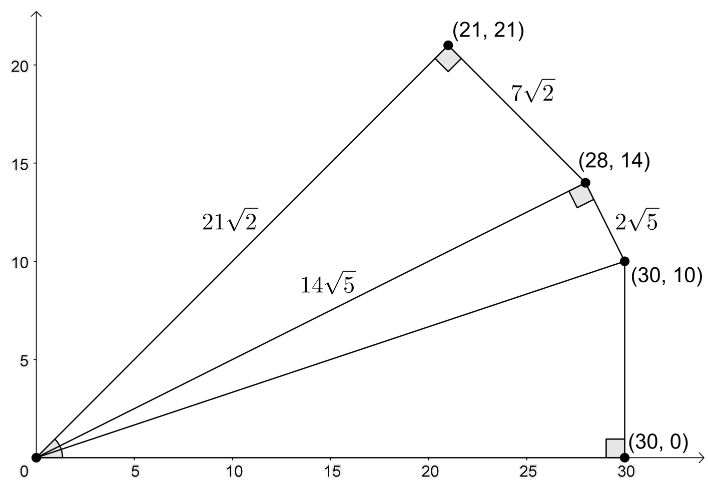
UNIVERSITY OF WISCONSIN
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #8

(Fall 2017)

Prove that $2\arctan\dfrac{1}{3}+\arctan\dfrac{1}{7}=\dfrac{\pi}{4}$.

Solution:



$\square$

Alternate solution:

Let $\theta=\arctan\frac{1}{3}$. Thus $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$ and $\tan\theta=\frac{1}{3}$. In fact, since $0<\tan\theta<\frac{\sqrt{3}}{3}$, we know that $0<\theta<\frac{\pi}{6}$. Likewise, if we write $\varphi=\arctan\frac{1}{7}$, then $\tan\varphi=\frac{1}{7}$ and $0<\varphi<\frac{\pi}{6}$. It follows that $2\theta+\varphi$ is in the interval $(0,\pi/2)$.

Moreover,

$$\tan(2\theta)=\frac{2\tan\theta}{1-\tan^2\theta}=\frac{2/3}{8/9}=\frac{3}{4},$$

so

$$\tan(2\theta+\varphi)=\frac{\tan 2\theta+\tan\varphi}{1-\tan 2\theta\tan\varphi}=\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\cdot\frac{1}{7}}\cdot\frac{28}{28}=\frac{21+4}{28-3}=1.$$

This shows that

$$2\theta+\varphi=\arctan(1),$$

so

$$2\arctan\frac{1}{3}+\arctan\frac{1}{7}=\frac{\pi}{4}.$$

Source: `http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fibpi.html`