University of Wisconsin
PLATTEVILLE
Department of Mathematics

Problem of the Week #6

(Fall 2017)

Evaluate the infinite product:

$$3^{1/3} \cdot 9^{1/9} \cdot 27^{1/27} \cdot 81^{1/81} \cdots$$

Solution:

$$\begin{aligned} P &= 3^{1/3} \cdot 9^{1/9} \cdot 27^{1/27} \cdot 81^{1/81} \cdots \\ P &= 3^{1/3} \cdot 3^{2/9} \cdot 3^{3/27} \cdot 3^{4/81} \cdots \\ \log_3 P &= \sum_{n=1}^{\infty} \frac{n}{3^n} \end{aligned}$$

Now choose your favorite way of applying the geometric series formula:

(1) Define a power series:

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} \frac{nx^{n-1}}{3^n} \\ \int f(x)\,dx &= C + \sum_{n=1}^{\infty} \left(\frac{x}{3}\right)^n \\ \int f(x)\,dx &= C + \frac{x/3}{1-(x/3)}, \quad x \in (-3,3) \\ f(x) &= \frac{3}{(3-x)^2} \\ \log_3 P = f(1) &= \frac{3}{4} \end{aligned}$$

(2)

$$\begin{aligned} \log_3 P &= \frac{1}{3} + \frac{2}{9} + \frac{3}{27} + \frac{4}{81} + \cdots \\ &= \frac{1}{3}\left(1 + \frac{2}{3} + \frac{3}{9} + \frac{4}{27} + \cdots\right) \\ &= \frac{1}{3}\left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots\right)^2 \\ &= \frac{1}{3}\left(\frac{1}{1-(1/3)}\right)^2 \\ &= \frac{3}{4} \end{aligned}$$

Either way, since $\log_3 P = \frac{3}{4}$, we have $P = 3^{3/4} = \boxed{\sqrt[4]{27}}$.