University of Wisconsin
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #5

(Fall 2017)

(a) Let f and g be functions for which $f \circ g = g$ and $g \circ f = f$. Prove: $f \circ f = f$ and $g \circ g = g$.

(b) Find distinct functions f and g with domain $\mathbb{R}$ for which $f \circ g = g$ and $g \circ f = f$.

Solution:

(a) $f \circ f = f \circ (g \circ f) = (f \circ g) \circ f = g \circ f = f$, and by symmetry $g \circ g = g$.

(b) There are many solutions. For one example, let $f(x) = \lfloor x \rfloor$ and $g(x) = \lceil x \rceil$.

This means that $f(x)$ is the greatest integer less than or equal to x, while $g(x)$ is the least integer greater than or equal to x.

Suppose $n \in \mathbb{Z}$. Then:

- $f(g(n)) = \lfloor \lceil n \rceil \rfloor = \lfloor n \rfloor = n = g(n)$,
- $g(f(n)) = \lceil \lfloor n \rfloor \rceil = \lceil n \rceil = n = f(n)$.

Suppose now that $x \in \mathbb{R}$ and $n < x < n + 1$. Then:

- $f(g(x)) = \lfloor \lceil x \rceil \rfloor = \lfloor n + 1 \rfloor = n + 1 = \lceil x \rceil = g(x)$,
- $g(f(x)) = \lceil \lfloor x \rfloor \rceil = \lceil n \rceil = n = \lfloor x \rfloor = f(x)$.

Thus $f \circ g = g$ and $g \circ f = f$, as desired.

Source: Swenson, Daniel. *Functions whose composition satisfies a simple rule* (preprint).