University of Wisconsin
PLATTEVILLE
Department of Mathematics

Problem of the Week #4

(Fall 2017)

Find all prime numbers p for which $71p+1$ is a perfect square.

Solution:
Let $71p+1=n^2$, with $n>0$. Then $71p=(n+1)(n-1)$, and since 71 is prime, either $71\,|\,(n+1)$ or $71\,|\,(n-1)$.
Suppose for the sake of contradiction that $71k=n+1$ for some $k\in\mathbb{N}$. Then $71p=71k(n-1)$, so $p=k(71k-2)$. Since p is prime and $k\in\mathbb{N}$, we must have $k=1$, so $p=69=3\cdot 23$ — but then p is not prime. $\Rightarrow\Leftarrow$
Thus $71k=n-1$ for some $k\in\mathbb{N}$. Thus $71p=(n+1)(71k)$, so $p=(71k+2)k$. Since p is prime and $k\in\mathbb{N}$, we must have $k=1$, and therefore $\boxed{p=73}$.

Source: Andreescu, Tito, and Jonathan Kane. *Purple Comet! Math Meet: The first ten years.* XYZ Press, 2013 (pp. 63, 281).