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# Problem of the Week #2
(Fall 2017)

Let $\{x_0, x_1, x_2, \dots\}$ be the sequence such that $x_0 = 1$ and (for $n \geq 0$)

$$x_{n+1} = \ln\left(e^{x_n} - x_n\right).$$

Prove that the infinite series $\sum_{k=0}^{\infty} x_k$ converges, and find its sum.

**Solution:**
For $n \geq 0$,

$$\begin{aligned} x_{n+1} &= \ln\left(e^{x_n} - x_n\right) \\ e^{x_{n+1}} &= e^{x_n} - x_n \\ x_n &= e^{x_n} - e^{x_{n+1}} \end{aligned}$$

Thus we obtain a telescoping partial sum: $x_0 + \cdots + x_n = e^{x_0} - e^{x_{n+1}}$. Suppose for a moment that $\{x_n\}$ converges to $L$. Then $L = \ln\left(e^L - L\right)$; solving, we get $L = 0$. Since $x_0 = 1$, we have

$$\sum_{k=0}^{\infty} x_k = e^{x_0} - e^L = e - 1.$$

It remains to prove that the sequence $\{x_n\}$ converges. To begin, we claim that $\{x_n\}$ is bounded below by 0. The claim is proved by induction on $n$, beginning with $x_0 = 1 > 0$. Assume $x_k > 0$. Let $f(x) = e^x - x$. Then $f'(x) = e^x - 1$, so $f'(x)$ is positive when $x > 0$. This means that $f(x)$ is increasing on $[0, \infty)$. In particular,

$$\begin{aligned} f(x_k) &> f(0) \\ e^{x_k} - x_k &> 1 \\ e^{x_{k+1}} &> 1 \\ x_{k+1} &> 0 \end{aligned}$$

which proves the claim.
Now that we know $x_n > 0$ for all $n$, we have $e^{x_n} - e^{x_{n+1}} > 0$. This means $x_{n+1} < x_n$: the sequence $\{x_n\}$ is decreasing. Since $\{x_n\}$ is decreasing and bounded below by 0, $\{x_n\}$ converges by the bounded monotone convergence theorem, and so

$$\sum_{k=0}^{\infty} x_k = \boxed{e - 1}.$$

**Source:** Kedlaya, Kiran, and Lenny Ng, "Solutions to the 77th William Lowell Putnam Mathematical Competition (Problem B1)."