UNIVERSITY OF WISCONSIN
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

PROBLEM OF THE WEEK #1

(Fall 2017)

Evaluate and simplify, for all $x \in [0, 2\pi]$:

$$F(x) = \int_{\sin x}^{\cos x} \sqrt{1 - t^2}\, dt$$

Solution:
By the first fundamental theorem of calculus and the chain rule,

$$\begin{aligned} F'(x) &= \frac{d}{dx}\left[\int_0^{\cos x} \sqrt{1-t^2}\, dt - \int_0^{\sin x} \sqrt{1-t^2}\, dt\right] \\ &= -(\sin x)\sqrt{1-\cos^2 x} - (\cos x)\sqrt{1-\sin^2 x} \\ &= -(\sin x)|\sin x| - (\cos x)|\cos x| \\ &= \begin{cases} -1 & x \text{ in quadrant I}, \\ \cos 2x & x \text{ in quadrant II}, \\ 1 & x \text{ in quadrant III}, \\ -\cos 2x & x \text{ in quadrant IV}. \end{cases} \\ F(x) &= \begin{cases} -x + C_1 & x \text{ in quadrant I}, \\ \frac{1}{2}\sin 2x + C_2 & x \text{ in quadrant II}, \\ x + C_3 & x \text{ in quadrant III}, \\ -\frac{1}{2}\sin 2x + C_4 & x \text{ in quadrant IV}. \end{cases} \end{aligned}$$

Note from the definition that $F(\pi/4) = \int_{\sqrt{2}/2}^{\sqrt{2}/2} \sqrt{1-t^2}\, dt = 0$, so $C_1 = \pi/4$. Likewise, $F(5\pi/4) = 0$, so $C_3 = -5\pi/4$.
Finally, since F is differentiable, F is continuous. Continuity at $x = \pi/2$ yields

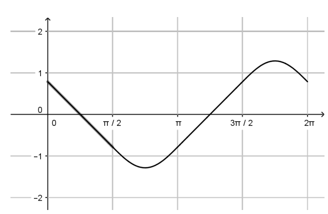
$$-\tfrac{\pi}{2} + \tfrac{\pi}{4} = \tfrac{1}{2}\sin\pi + C_2 \Rightarrow C_2 = -\tfrac{\pi}{4},$$

and from continuity at $x = 3\pi/2$, we get

$$\tfrac{3\pi}{2} - \tfrac{5\pi}{4} = -\tfrac{1}{2}\sin 3\pi + C_4 \Rightarrow C_4 = \tfrac{\pi}{4}.$$

Hence

$$\boxed{F(x) = \begin{cases} -x + \frac{\pi}{4} & x \text{ in quadrant I}, \\ \frac{1}{2}\sin 2x - \frac{\pi}{4} & x \text{ in quadrant II}, \\ x - \frac{5\pi}{4} & x \text{ in quadrant III}, \\ -\frac{1}{2}\sin 2x + \frac{\pi}{4} & x \text{ in quadrant IV}. \end{cases}}$$



Source: Maltenfort, Michael. "A Function Worth a Second Look." *College Mathematics Journal* **48**:1 (January 2017), 55-57.