UNIVERSITY OF WISCONSIN
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #9

(Fall 2016)

Find all perfect squares in the sequence $(9, 89, 889, 8889, 88889, \dots)$.

Solution:

The only perfect square in the sequence is 9.

Proof. Suppose $n^2 = 8\dots89$ is a d-digit number, with $d > 1$. Then

$$n^2 - 1 = 8\dots88 = 8(1\dots11) = 8\left(\frac{10^d - 1}{9}\right).$$

Since n^2 is odd, n is odd. Writing $n = 2k+1$, we get

$$\begin{aligned} 4k^2 + 4k &= 8\left(\frac{10^d - 1}{9}\right) \\ \frac{9k^2 + 9k + 2}{2} &= 10^d \\ (3k+2)(3k+1) &= 2^{d+1}5^d \end{aligned}$$

Suppose for the sake of contradiction that $2 \mid (3k+c)$ and $5 \mid (3k+c)$, where $c \in \{1,2\}$. Then $2 \nmid (3k+c\pm 1)$ and $5 \nmid (3k+c\pm 1)$. This means that $c = 2$ and $3k+1 = 1$ (since $3k+2 \neq 1$), so $k = 0$, and therefore $d = 0$, which is impossible.
Suppose now for the sake of contradiction that $3k+2 = 2^{d+1}$ and $3k+1 = 5^d$. Since $5^d < 2^{d+1}$, $\left(\frac{5}{2}\right)^d < 2$, so $d < 1$, which is impossible.
We conclude that $3k+2 = 5^d$ and $3k+1 = 2^{d+1}$. Since $d > 1$,

$$3k+2 = 5^d = (4+1)^d > 4^d + 1^d = 2^{2d} + 1 > 2^{d+1} + 1 = (3k+1) + 1 = 3k+2,$$

which is impossible. □

Source: Schoenecker, Kevin. "Crazy 8s (The Sandbox, Problem 346)." *Math Horizons* (November 2016), 30.

See also: Jaroma, J. H. "Triangular repunit — there is but 1." *Czech Math J.* **60** (2010), 1075.