University of Wisconsin
PLATTEVILLE
Department of Mathematics

Problem of the Week #8

(Fall 2016)

Evaluate:

$$\lim_{n\to\infty}\left[\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+4}}+\frac{1}{\sqrt{n^2+9}}+\frac{1}{\sqrt{n^2+16}}+\cdots+\frac{1}{\sqrt{n^2+n^2}}\right].$$

Solution:

$$\lim_{n\to\infty}\left[\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+4}}+\frac{1}{\sqrt{n^2+9}}+\cdots+\frac{1}{\sqrt{n^2+n^2}}\right]=\ln\left(1+\sqrt{2}\right).$$

Proof. Our goal is to evaluate

$$\begin{aligned} L &= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+k^2}} \\ &= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+k^2}} \\ &= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n\sqrt{1+(k/n)^2}} \\ &= \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\,\Delta x, \end{aligned}$$

where $f(x)=\frac{1}{\sqrt{1+x^2}}$, $\Delta x=\frac{1}{n}$, and $x_k = 0+k\,\Delta x$ for $1\le k\le n$. This is the right Riemann sum for the definite integral of $f(x)$ over the interval $[0,1]$, so

$$\begin{aligned} L &= \int_0^1 \frac{1}{\sqrt{1+x^2}}\,dx \\ &= \int_0^{\pi/4}\frac{1}{\sqrt{1+\tan^2\theta}}\sec^2\theta\,d\theta \qquad (x=\tan\theta) \\ &= \int_0^{\pi/4}\sec\theta\,d\theta \\ &= \ln\left|\sec\theta+\tan\theta\right|\Big|_{\theta=0}^{\pi/4} \\ &= \ln\left|\sqrt{2}+1\right|-\ln|1+0| \\ &= \ln\left(1+\sqrt{2}\right). \end{aligned}$$

□

Source: Kaczkowski, Stephen. "The Limiting Value of a Series with Exponential Terms." *Mathematics Magazine* **89**:4 (October 2016), p. 282.